A generalized Calderón formula for open-arc diffraction problems: theoretical considerations

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Abstract

We deal with the general problem of scattering by open-arcs in two-dimensional space. We show that this problem can be solved by means of certain second-kind integral equations of the form $\tilde{N}\tilde{S}[\varphi] = f$, where $\tilde{N}$ and $\tilde{S}$ are first-kind integral operators whose composition gives rise to a generalized Calderón formula of the form $\tilde{N}\tilde{S} = \tilde{J}_0^\tau + \tilde{K}$ in a *weighted, periodized* Sobolev space. (Here $\tilde{J}_0^\tau$ is a continuous and continuously invertible operator and $\tilde{K}$ is a compact operator.) The $\tilde{N}\tilde{S}$ formulation provides, for the first time, a second-kind integral equation for the open-arc scattering problem with Neumann boundary conditions. Numerical experiments show that, for both the Dirichlet and Neumann boundary conditions, our second-kind integral equations have spectra that are bounded away from zero and infinity as $k \to \infty$; to the authors' knowledge these are the first integral equations for these problems that possess this desirable property. This situation is in stark contrast with that arising from the related *classical* open-surface hypersingular and single-layer operators $\mathbf{N}$ and $\mathbf{S}$, whose composition $\mathbf{NS}$ maps, for example, the function $\phi = 1$ into a function that is not even square integrable. Our proofs rely on three main elements: 1) Algebraic manipulations enabled by the presence of integral weights; 2) Use of the classical result of continuity of the Cesàro operator; and 3) Explicit characterization of the point spectrum of $\tilde{J}_0^\tau$, which, interestingly, can be decomposed into the union of a countable set and an open set, both of which are tightly clustered around $-\frac{1}{4}$. As shown in a separate contribution, the new approach can be used to construct simple spectrally-accurate numerical solvers and, when used in conjunction with Krylov-subspace iterative solvers such as GMRES, it gives rise to dramatic reductions of iteration numbers vs. those required by other approaches.

1 Introduction

The field of Partial Differential Equations (PDEs) with boundary values prescribed on open surfaces has a long and important history, including significant contributions in the theory of diffraction by open screens, elasticity problems in solids containing cracks, and fluid flow past plates; solution to such problems impact significantly on present day technologies such as wireless transmission, electronics and photonics. From a mathematical point of view, besides techniques applicable to simple geometries, existing solution methods include special adaptations of finite-element and boundary-integral methods that account in some fashion for the singular character of the PDE solutions at edges. With much progress in the area over the last sixty years the field remains challenging: typically only low-frequency open-surface problems can be treated with any accuracy by previous approaches.

In this paper we focus on the problem of electromagnetic and acoustic scattering by open arcs. In particular, we introduce certain first-kind integral operators $\mathbf{N}_\omega$ and $\mathbf{S}_\omega$ whose composition gives rise, after appropriate change of periodic variables, to a generalized Calderón formula

$\tilde{N}\tilde{S} = \tilde{J}_0^\tau + \tilde{K}$—where $\tilde{J}_0^\tau$ is a continuous and continuously invertible operator and where $\tilde{K}$ is a compact operator—together with associated *second-kind open-surface integral equations* of the form $\tilde{N}\tilde{S}[\varphi] = f$. This approach enables, for the first time, treatment of open-arc scattering problems with Neumann boundary conditions by means of second kind equations. Further, a wide range of numerical experiments [12] indicate that, for both the Dirichlet and Neumann boundary conditions, our second-kind integral equations have spectra that are bounded away from zero and infinity as $k \to \infty$, and give rise to high accuracies and dramatic reductions of Krylov-subspace iteration numbers vs. those required by other approaches. These methods and results were first announced in [9]; succinct proofs of the open-arc Calderón formulae, further, were presented in [12].

Integral equation methods provide manifold advantages over other methodologies: they do not suffer from the well known pollution [2] and dispersion [20] errors characteristic of finite element and finite difference methods, they automatically enforce the condition of radiation at infinity (without use of absorbing boundary conditions), and they lend themselves to (typically iterative) acceleration techniques [4, 11, 31]—which can effectively take advantage of the reduced dimensionality arising from boundary-integral equations, even for problems involving very high-frequencies. Special difficulties inherent in open-surface boundary-integral formulations arise from the solution's edge singularity [17, 25, 34]. Such difficulties have typically been tackled by incorporating the singularity explicitly in both Galerkin [34, 35, 37] and Nyström [1, 21, 28] integral solvers; with one exception (introduced in the contributions [1, 21] and discussed below in some detail), in all of these cases integral equations of the first kind were used. While providing adequate discretizations of the problem, first-kind integral equation can be poorly conditioned and, for high-frequencies, they require large numbers of iterations and long computing times when accelerated iterative solvers as mentioned above are used.

(The literature on the singular behavior of open-arc solutions is quite rich and interesting from a historical perspective: it includes the early analysis [33], corrections [5, 6] to early contributions [3, 26], the well-known finite-energy condition introduced in [27], the integral equation formulation [25] and subsequent treatments for integral approaches for these problems, leading to the first regularity proof [34] and the comprehensive treatment [17] which establishes, in particular, that for C^∞ open surfaces with C^∞ edges, the integral equation solution of the Dirichlet (resp. Neumann) open-edge problems equals a C^∞ function times an unbounded (resp. bounded) canonical edge-singular function.)

As mentioned above, iterative solvers based on first-kind integral equations often require large numbers of iterations and long computing times. Attempts have been made over the years to obtain second-kind open-surface equations and, indeed, second kind equations for open surfaces were developed previously by exploiting the diagonal character of the logarithmic single layer, at least for the case of the *Dirichlet problem for the Laplace equation* [1, 21]. Unfortunately, as shown in [12], direct generalization of such approaches to high-frequency problems give rise to numbers of iterations that can in fact be much larger than those inherent in first-kind formulations. Efforts were also made to obtain second kind equations on the basis of the well known Calderón formula. The Calderón identity relates the classical single-layer and hypersingular operators $\mathbf{S}_c$ and $\mathbf{N}_c$ that are typically associated with the Dirichlet and Neumann problems on a closed surface Γ_c: for such closed surfaces the Calderón formula reads $\mathbf{N}_c\mathbf{S}_c = -\mathbf{I}/4 + \mathbf{K}_c$, where $\mathbf{K}_c$ is a compact operator in a suitable Sobolev space. Attempts to extend this idea to open surfaces were pursued in [14, 30]. As first shown in [30], there indeed exists a related identity for the corresponding single-layer and hypersingular operators $\mathbf{N}$ and $\mathbf{S}$ on an open surface Γ. As in the closed-surface case, we have $\mathbf{NS} = -\mathbf{I}/4 + \mathbf{K}$; unfortunately, however, a useful functional setting for the operator $\mathbf{NS}$ does not appear to exist ($\mathbf{K}$ is not compact). As shown in Appendix B, for example, the composition $\mathbf{NS}$ maps the constant function $\varphi = 1$ on an open surface into a function that tends to infinity at the boundary of Γ like $1/d$, where d denotes the distance to the curve edge (see Appendix B). In particular, the formulation $\mathbf{NS}$ cannot be placed in the functional framework put forth in [34, 35, 37] and embodied by equations (10), (11) and Definition 1 below: a function with $1/d$ edge asymptotics is not an element of $H^{-\frac{1}{2}}(\Gamma)$.

In view of the aforementioned regularity result [17]—which, can be fully exploited numerically through use of Chebyshev expansions [1, 28] in two-dimensions and appropriate extensions [13] of the high-order integration methods [11] in three-dimensions—, and on account of the results of this paper, use of the combination $N_\omega S_\omega$ enables low-iteration-number, second-kind, super-algebraically accurate solution of open surface scattering problems—and thus gives rise to a highly efficient numerical solver for open surface scattering problems in two and three dimensions; see [12, 13].

This paper is organized as follows. After introduction of necessary notations and preliminaries, the main result of this paper, Theorem 1, is stated in Section 2. Section 3 contains the main elements of the theory leading to a proof of Theorem 1. Necessary uniqueness and regularity results for the single-layer and hyper-singular weighted operators under a cosine change of variables, which have mostly been known for a number of years (see [32, Ch. 11] and the extensive literature cited therein) are presented in Section 4; the inclusion of these results renders our text essentially self contained, and it establishes a direct link between our context and that represented by references [34, 35, 37]. Building up on constructions presented in earlier sections, the proof of Theorem 1 is given in Section 5. Two appendices complete our contribution: Appendix A presents a version adequate to our context of a known expression linking the hypersingular operator and an integro-differential operator containing only tangential derivatives; Appendix B, finally, demonstrates that the image of the composition $\mathbf{NS}$ of the un-weighted operators $\mathbf{N}$ and $\mathbf{S}$ is not contained in $H^{-\frac{1}{2}}$.

2 Preliminaries

Throughout this paper Γ is assumed to be a smooth open arc in two dimensional space.

2.1 Background

As is well known [32, 35, 37], the Dirichlet and Neumann boundary-value problems for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{outside} \quad \Gamma, \quad u|_\Gamma = f, \quad f \in H^{\frac{1}{2}}(\Gamma) \qquad \text{(Dirichlet)} \\ \Delta v + k^2 v = 0 \quad \text{outside} \quad \Gamma, \quad \frac{\partial v}{\partial n}|_\Gamma = g, \quad g \in H^{-\frac{1}{2}}(\Gamma) \qquad \text{(Neumann)} \end{cases} \tag{1}$$

admit unique radiating solutions $u, v \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$ which can be expressed in terms of single- and double-layer potentials, respectively:

$$u(\mathbf{r}) = \int_\Gamma G_k(\mathbf{r}, \mathbf{r}')\mu(\mathbf{r}')d\ell' \tag{2}$$

and

$$v(\mathbf{r}) = \int_\Gamma \frac{\partial G_k(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}}\nu(\mathbf{r}')d\ell' \tag{3}$$

for $\mathbf{r}$ outside Γ. Here $\mathbf{n}_{\mathbf{r}'}$ is a unit vector normal to Γ at the point $\mathbf{r}' \in \Gamma$ (we assume, as we may, that $\mathbf{n}_{\mathbf{r}'}$ is a smooth function of $\mathbf{r}' \in \Gamma$), and, letting H_0^1 denote the Hankel function,

$$G_k(\mathbf{r},\mathbf{r}') = \begin{cases} \frac{i}{4}H_0^1(k|\mathbf{r}-\mathbf{r}'|), & k>0 \\ -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|, & k=0 \end{cases}, \tag{4}$$

and

$$\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}} = \mathbf{n}_{\mathbf{r}'} \cdot \nabla_{\mathbf{r}'} G_k(\mathbf{r},\mathbf{r}'). \tag{5}$$

Denoting by $\mathbf{S}$ and $\mathbf{N}$ the single-layer and hypersingular operators

$$\mathbf{S}[\mu](\mathbf{r}) = \int_\Gamma G_k(\mathbf{r},\mathbf{r}')\mu(\mathbf{r}')d\ell' \quad , \quad \mathbf{r} \in \Gamma, \tag{6}$$

and

$$\begin{aligned} \mathbf{N}[\nu](\mathbf{r}) &= \frac{\partial}{\partial \mathbf{n}_{\mathbf{r}}} \int_\Gamma \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}} \nu(\mathbf{r}')d\ell' \\ &\stackrel{\text{def}}{=} \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_\Gamma \frac{\partial G_k(\mathbf{r}+z\mathbf{n}_{\mathbf{r}},\mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}} \nu(\mathbf{r}')d\ell' \quad , \quad \mathbf{r} \in \Gamma, \end{aligned} \tag{7}$$

the densities μ and ν are the unique solutions of the first kind integral equations

$$\mathbf{S}[\mu] = f \tag{8}$$

and

$$\mathbf{N}[\nu] = g. \tag{9}$$

As shown in [34, 35, 37], the operators $\mathbf{S}$ and $\mathbf{N}$ define bounded and continuously invertible mappings

$$\mathbf{S}: \ \tilde{H}^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad \text{and} \tag{10}$$

$$\mathbf{N}: \ \tilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \tag{11}$$

where for $s \in \mathbb{R}$, the space $\tilde{H}^s(\Gamma)$ is defined below.

Definition 1 *Let G_1 be a domain in the plane, with a smooth boundary $\dot{G}_1$, let $s \in \mathbb{R}$, and assume $\dot{G}_1$ contains the smooth open curve Γ. The Sobolev space $\tilde{H}^s(\Gamma)$ is defined as the set of all elements $f \in H^s(\dot{G}_1)$ satisfying* $\mathrm{supp}(f) \subseteq \overline{\Gamma}$.

Remark 1 *As is well known [7, Corollary 2.7], the inverse L^{-1} of a continuous and invertible (one-to-one and surjective) operator L between two Banach spaces (and, in particular, between two Hilbert spaces such as the Sobolev spaces considered in this text) is also continuous. In view of this fact, above and throughout this text the terms "invertible continuous operator", "invertible bounded operator", "bicontinuous operator", "continuous operator with continuous inverse", etc, are used as interchangeable synonyms.*

The mapping results (10), (11) provide an extension of classical closed-surfaces results: for a closed Lipschitz surface Γ_c and for any $s \in \mathbb{R}$, the closed-surface single-layer and hypersingular operators $\mathbf{S}_c$ and $\mathbf{N}_c$ define bounded mappings

$$\mathbf{S}_c: \ H^s(\Gamma_c) \to H^{s+1}(\Gamma_c), \tag{12}$$

$$\mathbf{N}_c: \; H^{s+1}(\Gamma_c) \to H^s(\Gamma_c), \tag{13}$$

see e.g. [16, 22, 29]. Additionally, the closed-surface potentials satisfy the classical Calderón relation

$$\mathbf{N}_c \mathbf{S}_c = -\frac{\mathbf{I}}{4} + \mathbf{K}_c \tag{14}$$

in $H^s(\Gamma_c)$, where $\mathbf{K}_c$ is a compact operator.

While (12) and (13) do not apply to open surfaces, the solutions μ and ν of the open-surface integral equations (8) and (9) enjoy significant regularity properties. In particular, letting $d = d(\mathbf{r})$ denote any non-negative smooth function defined on Γ which for $\mathbf{r}$ in a neighborhood of each end-point equals the Euclidean distance from $\mathbf{r}$ to the corresponding end-point, and letting ω denote any function defined on Γ such that $\omega/\sqrt{d}$ is C^∞ up to the endpoints, the recent results [17] establish that if the arc Γ and the right-hand-side functions f and g in (8) and (9) are infinitely differentiable we have

$$\mu = \frac{\alpha}{\omega} \tag{15}$$

and

$$\nu = \beta \cdot \omega, \tag{16}$$

where α and β are C^∞ functions throughout Γ. The singular behavior in these solutions is thus fully characterized by the factors $d^{1/2}$ and $d^{-1/2}$ in equations (15) and (16), respectively.

2.2 Generalized Calderón Formula

In view of equations (15) and (16), for any non-vanishing function $\omega(\mathbf{r}) > 0$ such that $\omega/\sqrt{d}$ is C^∞ up to the endpoints of Γ, we define the weighted operators

$$\mathbf{S}_\omega[\alpha] = \mathbf{S}\left[\frac{\alpha}{\omega}\right] \tag{17}$$

and

$$\mathbf{N}_\omega[\beta] = \mathbf{N}\left[\beta \cdot \omega\right], \tag{18}$$

and we consider the weighted versions

$$\mathbf{S}_\omega[\alpha] = f \tag{19}$$

and

$$\mathbf{N}_\omega[\beta] = g \tag{20}$$

of the integral equations (8) and (9); clearly, in view of the discussion of the previous section, for smooth Γ and smooth right-hand-sides f and g, the solutions α and β of (19) and (20) are smooth up to the endpoints of Γ.

Without loss of generality we use a smooth parametrization $\mathbf{r}(t) = (x(t), y(t))$ of Γ defined in the interval $[-1, 1]$, for which $\tau(t) = |\frac{d\mathbf{r}(t)}{dt}|$ is never zero. For definiteness and simplicity, throughout the rest of the paper we select ω, as we may, in such a way that

$$\omega(\mathbf{r}(t)) = \sqrt{1 - t^2}. \tag{21}$$

The operators $\mathbf{S}_\omega$ and $\mathbf{N}_\omega$ thus induce the parameter-space operators

$$S_\omega[\varphi](t) = \int_{-1}^{1} G_k\left(\mathbf{r}(t), \mathbf{r}(t')\right) \frac{\varphi(t')}{\sqrt{1 - t'^2}}\, \tau(t')dt', \tag{22}$$

and

$$N_\omega[\psi](t) = \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_{-1}^{1} \frac{\partial}{\partial \mathbf{n}_{\mathbf{r}(t')}} G_k\left(\mathbf{r}(t) + z\mathbf{n}_{\mathbf{r}(t)}, \mathbf{r}(t')\right) \psi(t')\tau(t')\sqrt{1-t'^2}dt'; \tag{23}$$

defined on functions φ and ψ of the variable t, $-1 \le t \le 1$; clearly, for $\varphi(t) = \alpha(\mathbf{r}(t))$ and $\psi(t) = \beta(\mathbf{r}(t))$ we have

$$\mathbf{S}_\omega[\alpha](\mathbf{r}(t)) = S_\omega[\varphi](t) \tag{24}$$

and

$$\mathbf{N}_\omega[\beta](\mathbf{r}(t)) = N_\omega[\psi](t). \tag{25}$$

In order to proceed we further transform our integral operators: using the changes of variables $t = \cos\theta$ and $t' = \cos\theta'$ and, defining $\mathbf{n}_\theta = \mathbf{n}_{\mathbf{r}(\cos\theta)}$ and using (24) and (25), we re-express equations (19) and (20) in the forms

$$\tilde{S}[\tilde{\varphi}] = \tilde{f} \tag{26}$$

and

$$\tilde{N}[\tilde{\psi}] = \tilde{g}, \tag{27}$$

where $\tilde{S}$ and $\tilde{N}$ denote the operators

$$\tilde{S}[\gamma](\theta) = \int_0^\pi G_k(\mathbf{r}(\cos\theta), \mathbf{r}(\cos\theta'))\gamma(\theta')\tau(\cos\theta')d\theta' \tag{28}$$

and

$$\tilde{N}[\gamma](\theta) = \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^\pi \frac{\partial}{\partial \mathbf{n}_{\theta'}} G_k(\mathbf{r}(\cos\theta) + z\mathbf{n}_\theta, \mathbf{r}(\cos\theta'))\gamma(\theta')\tau(\cos\theta')\sin^2\theta' d\theta', \tag{29}$$

and where

$$\tilde{f}(\theta) = f(\mathbf{r}(\cos\theta)) \quad , \quad \tilde{g}(\theta) = g(\mathbf{r}(\cos\theta)); \tag{30}$$

clearly, the solutions of equations (22)-(27) are related by

$$\tilde{\varphi}(\theta) = \varphi(\cos\theta) \quad , \quad \tilde{\psi}(\theta) = \psi(\cos\theta). \tag{31}$$

In view of the symmetries induced by the $\cos\theta$ dependence in equations (28) through (30), it is natural to study the properties of these operators and equations in appropriate Sobolev spaces $H^s_e(2\pi)$ of 2π periodic and even functions defined below; cf. [10, 38].

Definition 2 *Let $s \in \mathbb{R}$. The Sobolev space $H^s_e(2\pi)$ is defined as the completion of the space of infinitely differentiable 2π-periodic and even functions defined in the real line with respect to the norm*

$$\|v\|^2_{H^e_s(2\pi)} = |a_0|^2 + 2\sum_{m=1}^{\infty} m^{2s}|a_m|^2, \tag{32}$$

where a_m denotes the m-th cosine coefficient of v:

$$v(\theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta). \tag{33}$$

Clearly the set $\{\cos(n\theta) : n \in \mathbb{N}\}$ is a basis of the Hilbert space $H^s_e(2\pi)$ for all s.

For notational convenience we also introduce corresponding discrete sequence spaces h^s, $s \ge 0$ and ℓ^2.

Definition 3 *Let $s \geq 0$. The Hilbert space h^s is defined as the space of all sequences $a = (a_n)_{n\in\mathbb{N}}$ of complex numbers with finite norm $||a||_{h^s} < \infty$, with the discrete s-norm $||\cdot||_{h^s}$ defined by*

$$||a||_{h^s}^2 = |a_0|^2 + 2\sum_{n=1}^{\infty} |a_n|^2 n^{2s}. \tag{34}$$

and with the natural associated scalar product. We also define $\ell^2 = h^0$.

The main purpose of this paper is to establish the following theorem.

Theorem 1 *The composition $\tilde{N}\tilde{S}$ defines a bicontinuous operator from $H_e^s(2\pi)$ to $H_e^s(2\pi)$ for all $s > 0$. Further, this operator satisfies a generalized Calderón formula*

$$\tilde{N}\tilde{S} = \tilde{J}_0^\tau + \tilde{K}, \tag{35}$$

where $\tilde{K} : H_e^s(2\pi) \to H_e^s(2\pi)$ is a compact operator, and where $\tilde{J}_0^\tau : H_e^s(2\pi) \to H_e^s(2\pi)$ is a bicontinuous operator, independent of k, with point spectrum equal to the union of the discrete set $\Lambda_\infty = \{\lambda_0 = -\frac{\ln 2}{4}, \quad \lambda_n = -\frac{1}{4} - \frac{1}{4n} : n > 0\}$ and a certain open set set Λ_s which is bounded away from zero and infinity. The sets Λ_s are nested, they form a decreasing sequence, and they satisfy $\bigcap_{s>0} \bar{\Lambda}_s = \{-\frac{1}{4}\}$, where $\bar{\Lambda}_s$ denotes the closure of Λ_s. In addition, the operators

$$\tilde{S} : \; H_e^s(2\pi) \to H_e^{s+1}(2\pi) \quad \text{and} \tag{36}$$

$$\tilde{N} : \; H_e^{s+1}(2\pi) \to H_e^s(2\pi) \tag{37}$$

are bicontinuous.

We thus see that, through introduction of the weight ω and use of spaces of even and 2π periodic functions, a picture emerges for the open-surface case that resembles closely the one found for closed-surface configurations: the generalized Calderón relation (35) is analogous to the Calderón formula (14), and mapping properties in terms of the complete range of Sobolev spaces are recovered for $\tilde{S}$ and $\tilde{N}$, in a close analogy to the framework embodied by equations (12) and (13).

In the remainder of this paper we present a proof of Theorem 1. This proof is based on a number of elements, the first one of which, presented in Section 3, concerns the operator $\tilde{J}_0^\tau$ in (35)—which corresponds, in fact, to the zero-frequency/straight-arc version of Theorem 1.

3 Straight arc at zero frequency: operators $\tilde{J}_0$ and $\tilde{J}_0^\tau$

3.1 Preliminary properties of the operators $\tilde{S}_0$, $\tilde{N}_0$ and other related operators

In the case in which Γ is the straight-arc $[-1, 1]$ and $k = 0$, $\tilde{S}$ reduces to Symm's operator [10, 38]

$$\tilde{S}_0[\tilde{\varphi}](\theta) = -\frac{1}{2\pi}\int_0^\pi \ln|\cos\theta - \cos\theta'|\tilde{\varphi}(\theta)d\theta, \tag{38}$$

for which the following lemma holds

Lemma 1 *The operator $\tilde{S}_0$ maps $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$, and*

$$\tilde{S}_0 : \quad H_e^s(2\pi) \to H_e^{s+1}(2\pi) \quad \text{is bicontinuous for all } s \geq 0. \tag{39}$$

Proof. It follows from the weak singularity of the kernel in equation (38) that

$$\tilde{S}_0 : \quad H_e^0(2\pi) \to H_e^0(2\pi) \quad \text{is a continuous operator.} \tag{40}$$

Furthermore, taking into account the well documented diagonal property [24]

$$\tilde{S}_0[e_n] = \lambda_n e_n, \quad \lambda_n = \begin{cases} \frac{\ln 2}{2} & n = 0 \\ \frac{1}{2n}, & n \geq 1 \end{cases} \tag{41}$$

of Symm's operator in the basis $\{e_n : n \geq 0\}$ of $H_e^s(2\pi)$ defined by

$$e_n(\theta) = \cos n\theta, \tag{42}$$

we see that, for every basis element $e_n, n \geq 0$, the operator $\tilde{S}_0$ coincides with the diagonal operator defined by

$$W[f] = \sum_{n \geq 0} \lambda_n f_n e_n \quad \text{for} \quad f = \sum_{n \geq 0} f_n e_n \in H_e^s(2\pi). \tag{43}$$

Clearly the operator $W : \quad H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ is bicontinuous for all $s \geq 0$, and it is in particular a continuous operator from $H_e^0(2\pi)$ into $H_e^0(2\pi)$. The continuous operators $\tilde{S}_0$ and W thus coincide on the dense set $\{e_n\}$ of $H_e^0(2\pi)$, and they are therefore equal throughout $H_e^0(2\pi)$. It follows that $\tilde{S}_0 = W$ maps $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$ bicontinuously, and the proof is complete. □

The corresponding zero-frequency straight-arc version $\tilde{N}_0$ of the operator $\tilde{N}$, in turn, is given by

$$\tilde{N}_0[\tilde{\psi}](\theta) = \frac{1}{4\pi} \lim_{z \to 0} \frac{\partial^2}{\partial z^2} \int_0^\pi \ln|(\cos\theta - \cos\theta')^2 + z^2|\tilde{\psi}(\theta') \sin^2\theta' d\theta', \tag{44}$$

which, following [15, 22, 28] we express in the form

$$\tilde{N}_0 = \tilde{D}_0 \tilde{S}_0 \tilde{T}_0 \tag{45}$$

where

$$\tilde{D}_0[\tilde{\varphi}](\theta) = \frac{1}{\sin\theta} \frac{d\tilde{\varphi}(\theta)}{d\theta} \tag{46}$$

and

$$\tilde{T}_0[\tilde{\varphi}](\theta) = \frac{d}{d\theta}\left(\tilde{\varphi}(\theta) \sin\theta\right). \tag{47}$$

(The general curved-arc arbitrary-frequency version of this relation is presented in Lemma 13 below and, for the sake of completeness, a derivation of the general relation is provided in Appendix A.)

Note that, in contrast with the closed-arc case [22, p. 117], the expressions (45) through (47) contain the vanishing factor $\sin\theta$ and the singular factor $1/\sin\theta$; in particular, for example, it is not immediately clear that the operator $\tilde{N}_0$ maps $H_e^{s+1}(2\pi)$ into $H_e^s(2\pi)$. This result is presented in Corollary 3. In preparation for our proofs of that and other straight-arc zero-frequency results in the following sections, in the remainder of this section we establish a preliminary continuity result for the operator $\tilde{D}_0$.

Lemma 2 *The operator $\tilde{D}_0$ defines a bounded mapping from $H_e^2(2\pi)$ into $H_e^0(2\pi)$.*

Proof. Let $\tilde{\varphi} = \sum_{n=0}^{\infty} \tilde{\varphi}_n e_n$ be an element of $H_e^2(2\pi)$. We assume at first that $\tilde{\varphi}_{2p+1} = 0$ for all integers $p \geq 0$. Let $P > 0$ and $\tilde{\varphi}^P = \sum_{p=0}^{P} \tilde{\varphi}_{2p} e_{2p}$; clearly $\tilde{\varphi}^P$ converges to $\tilde{\varphi}$ in $H_e^2(2\pi)$ as $P \to \infty$. We have

$$\tilde{D}_0[\tilde{\varphi}^P](\theta) = -\sum_{p=0}^{P} 2p\tilde{\varphi}_{2p} \frac{\sin(2p\theta)}{\sin\theta} \tag{48}$$

In view of the identity

$$\frac{\sin(n+1)\theta}{\sin\theta} = \begin{cases} \sum\limits_{k=0}^{p} (2-\delta_{0k})\cos 2k\theta, & n = 2p \\ 2\sum\limits_{k=0}^{p} \cos(2k+1)\theta, & n = 2p+1, \end{cases} \tag{49}$$

(which expressed in terms of Chebyshev polynomials of the first and second kind is given e.g. in equation (40) [18, p. 187] and problem 3 in [24, p. 36]), we obtain

$$\tilde{D}_0[\tilde{\varphi}^P] = -2\sum_{k=1}^{P} \left(\sum_{p=k}^{\infty} 2p\tilde{\varphi}_{2p}^P \right) e_{2k-1} \tag{50}$$

where

$$\tilde{\varphi}_{2p}^P = \begin{cases} \tilde{\varphi}_{2p}^P, & p \leq P \\ 0, & p > P \end{cases}.$$

The quantity in parenthesis on the right-hand-side of equation (50) can be expressed in terms of the adjoint C^* of the discrete Cesàro operator C, where C and C^* are given by

$$C[g](n) = \frac{1}{n+1}\sum_{k=0}^{n} g_k, \quad C^*[g](k) = \sum_{p=k}^{\infty} \frac{g_p}{p+1}; \tag{51}$$

as there follows from [8], C and C^* define bounded operators from ℓ^2 into ℓ^2. We thus re-express equation (50) as

$$\tilde{D}_0[\tilde{\varphi}^P] = -2\sum_{k=1}^{P} C^*[g^P](k) e_{2k-1}, \tag{52}$$

where the sequence g^P is given by $g_p^P = 2p(p+1)\tilde{\varphi}_{2p}^P$. Clearly g^P is an element of ℓ^2 and we have

$$||g^P||_{\ell^2} = \left(\sum_{p=1}^{P} |g_p^P|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{p=1}^{P} (2p)^4 |\tilde{\varphi}_{2p}^P|^2 \right)^{\frac{1}{2}} \leq 2^{1/2} ||\tilde{\varphi}^P||_{H_e^2(2\pi)}. \tag{53}$$

In view of the boundedness of C^* as an operator from ℓ^2 into ℓ^2 we obtain

$$||\tilde{D}_0[\tilde{\varphi}^P]||_{H_e^0(2\pi)} \leq 2||C^*||_{\ell^2} ||g^P||_{\ell^2}, \tag{54}$$

and thus, in view of (53),

$$||\tilde{D}_0[\tilde{\varphi}^P]||_{H_e^0(2\pi)} \leq 2^{3/2} ||C^*||_{\ell^2} ||\tilde{\varphi}^P||_{H_e^2(2\pi)}. \tag{55}$$

A similar manipulation on odd-termed sequences can be performed and, in all, it follows that for any $P > 0$ and any element $\tilde{\varphi} = \sum_{n=0}^{\infty} \tilde{\varphi}_n e_n \in H_e^2(2\pi)$ the corresponding finite-term truncation $\tilde{\varphi}^P = \sum_{n=0}^{P} \tilde{\varphi}_n e_n$ satisfies

$$||\tilde{D}_0[\tilde{\varphi}^P]||_{H_e^0(2\pi)} \leq K||\tilde{\varphi}^P||_{H_e^2(2\pi)} \tag{56}$$

for some constant K which does not depend on P.

Since $\tilde{\varphi}^P$ converges in $H_e^2(2\pi)$ to $\tilde{\varphi}$ as $P \to \infty$, it follows that $\tilde{\varphi}^P$ is a Cauchy sequence in $H_e^2(2\pi)$:

$$||\tilde{\varphi}^P - \tilde{\varphi}^Q||_{H_e^2(2\pi)} \to 0 \quad \text{as} \quad P, Q \to \infty. \tag{57}$$

Now, $(\tilde{\varphi}^P - \tilde{\varphi}^Q)$ can be viewed as a finite-term truncation of an element of $H_e^2(2\pi)$ and, thus, the estimate (56) applies to it: we obtain

$$||\tilde{D}_0[\tilde{\varphi}^P] - \tilde{D}_0[\tilde{\varphi}^Q]||_{H_e^0(2\pi)} \leq K||\tilde{\varphi}^P - \tilde{\varphi}^Q||_{H_e^2(2\pi)}. \tag{58}$$

Equations (57)-(58) show that $\tilde{D}_0[\tilde{\varphi}^P]$ is a Cauchy sequence in $H_e^0(2\pi)$. It follows that the sequence $\tilde{D}_0[\tilde{\varphi}^P]$ converges in that space as $P \to \infty$ and, in particular, that $\tilde{D}_0[\tilde{\varphi}]$ is an element of $H_e^0(2\pi)$. Taking limit as $P \to \infty$ in equation (56), finally, yields the inequality

$$||\tilde{D}_0[\tilde{\varphi}]||_{H_e^0(2\pi)} \leq K||\tilde{\varphi}||_{H_e^2(2\pi)}$$

which establishes the needed boundedness of the operator $\tilde{D}_0$. The proof is now complete. $\square$

Corollary 1 *The operator $\tilde{N}_0$ defines a bounded mapping from $H_e^2(2\pi)$ into $H_e^0(2\pi)$.*

Proof. The proof follows from Lemma 2, the decomposition (45), the continuity of $\tilde{S}_0$ established in (39), and the easily verified observation that $\tilde{T}_0$ defines a continuous operator from H_e^{s+1} and H_e^s:

$$\tilde{T}_0: \quad H_e^{s+1}(2\pi) \to H_e^s(2\pi). \tag{59}$$

$\square$

A preliminary boundedness result for the composite operator $\tilde{J}_0 = \tilde{N}_0\tilde{S}_0$ follows from this Corollary.

Corollary 2 *The straight-arc zero-frequency version of the composite operator $\tilde{N}\tilde{S}$, which is given by*

$$\tilde{J}_0 = \tilde{N}_0\tilde{S}_0, \tag{60}$$

defines a bounded operator from $H_e^1(2\pi)$ into $H_e^0(2\pi)$.

In the following section we show that, as stated in Theorem 1 for the related operator $\tilde{J}_0^\tau$ (cf. Section 3.5), not only does $\tilde{J}_0$ define a continuous operator from $H_e^1(2\pi)$ into $H_e^0(2\pi)$ (Corollary 2): $\tilde{J}_0$ can also be viewed as a continuous operator from $H_e^s(2\pi)$ into $H_e^s(2\pi)$ for all $s \geq 0$.

3.2 Boundedness of $\tilde{J}_0$ in H_e^s and link with the continuous Cesàro operator

The continuity proof presented in this section is based in part on the following lemma, whose proof relies on use of a certain operator $\tilde{C}$ related to the *continuous* Cesàro operator. (Note that the constructions in Section 3.1 invoke properties of the *discrete* Cesàro operator, instead).

Lemma 3 *For all $s \geq 0$ the integral operator*

$$\tilde{C}[\tilde{\varphi}](\theta) = \frac{\theta(\pi-\theta)}{\pi \sin\theta}\left[\frac{1}{\theta}\int_0^\theta \tilde{\varphi}(u)du - \frac{1}{\pi-\theta}\int_\theta^\pi \tilde{\varphi}(y)du\right] \tag{61}$$

maps $H_e^s(2\pi)$ continuously to itself. Furthermore,

$$\tilde{C}[e_n](\theta) = \begin{cases} 0 & \textit{for } n = 0 \\ \frac{\sin n\theta}{n \sin\theta} & \textit{for } n > 0, \end{cases} \tag{62}$$

for all $n \geq 0$

Proof. The relation (62) results from simple manipulations. The integral operator on the right-hand side of equation (61) in turn, can be expressed in terms of the continuous Cesàro operator

$$C[f](x) = \frac{1}{x}\int_0^x f(u)du = \int_0^1 f(xu)du. \tag{63}$$

As is known [8], C is a bounded operator from $L^2[0,b]$ into $L^2[0,b]$ (the space of square-integrable functions over $[0,b]$) for all $b > 0$. In view of the relations (62) it follows that the operator $\tilde{C}$ can be extended in a unique manner as a bounded operator from $H^0(2\pi)$ to $H^0(2\pi)$. Taking into account equation (63), further, for each $f \in C_0^\infty[0,b]$, $m \in \mathbb{N}$ and $x \in [0,b]$ we obtain

$$\left|\frac{\partial^m C[f](x)}{\partial x^m}\right|^2 \leq \left(\int_0^1 \left|u^m f^{(m)}(xu)\right| du\right)^2 \leq \left(\int_0^1 \left|f^{(m)}(xu)\right| du\right)^2 = \left(C\left[g\right](x)\right)^2 \tag{64}$$

where $g = \left|f^{(m)}\right|$. Integrating this inequality with respect to x and taking into account the boundedness of C as an operator from L^2 to L^2 there results

$$\int_0^{2\pi} \left|\frac{\partial^m C[f](x)}{\partial x^m}\right|^2 dx \leq M||f^{(m)}||^2_{L_2[0,2\pi]} \tag{65}$$

for some constant M. It follows easily from this inequality that $\tilde{C}$ is a continuous operator from $H_e^m(2\pi)$ into $H_e^m(2\pi)$ for all non-negative integers m. Letting $H^m(2\pi)$ be the space of 2π periodic functions whose derivatives of order k are square integrable in any bounded set of the line for all integers $k \leq m$ (c.f. [22]) we see that $\tilde{C}$ equals the restriction to $H_e^m(2\pi)$ of some continuous operator $\tilde{P} : H^m(2\pi) \to H^m(2\pi)$: we may simply take, for example, $\tilde{P}$ to equal $\tilde{C}$ on the subspace of even functions and to equal 0 on the space of odd functions. In view of the Sobolev interpolation result (see e.g. [22, Theorem 8.13]), $\tilde{P}$ defines a continuous operator from $H^s(2\pi)$ to $H^s(2\pi)$ for all $s \geq 0$, and thus, by restriction of $\tilde{P}$ to the susbspace of even and period functions we see that $\tilde{C}$ is a continuous operator from $H_e^s(2\pi)$ to $H_e^s(2\pi)$ for all $s \geq 0$. □

Our main result concerning the operator $\tilde{J}_0$ is given in the following lemma.

Lemma 4 *The composition $\tilde{J}_0 = \tilde{N}_0\tilde{S}_0$ defines a bounded operator from $H_e^s(2\pi)$ into $H_e^s(2\pi)$ for all $s \geq 0$.*

Proof. We first evaluate the action of $\tilde{J}_0$ on the basis $\{e_n : n \geq 0\}$. The case $n = 0$ is straightforward: in view of (41) and (45) we have

$$\tilde{J}_0[e_0](\theta) = -\frac{\ln 2}{4}. \tag{66}$$

For $n \geq 0$, in turn, expanding (47) we obtain

$$\begin{aligned}\tilde{T}_0[e_n](\theta) &= \cos\theta\cos n\theta - n\sin n\theta\sin\theta \\ &= \frac{\cos(n+1)\theta + \cos(n-1)\theta}{2} + n\frac{\cos(n+1)\theta - \cos(n-1)\theta}{2}\end{aligned} \tag{67}$$

which, for $n \geq 2$, in view of (41) yields, upon application of $\tilde{S}_0$,

$$\tilde{S}_0\tilde{T}_0[e_n](\theta) = \frac{\cos(n+1)\theta}{4(n+1)} + \frac{\cos(n-1)\theta}{4(n-1)} + n\left(\frac{\cos(n+1)\theta}{4(n+1)} - \frac{\cos(n-1)\theta}{4(n-1)}\right) \quad , \quad (n \geq 2). \tag{68}$$

In view of (45)-(46), for $n \geq 2$ we thus obtain the relation

$$\tilde{N}_0[e_n](\theta) = -\cos\theta\frac{\sin n\theta}{2\sin\theta} - \frac{n}{2}\cos n\theta, \tag{69}$$

which, as it is easily verified, also holds for $n = 1$. Using this relation in conjunction with (41) and (66) it follows that

$$\tilde{J}_0[e_n](\theta) = \begin{cases} -\frac{\ln 2}{4}, & n = 0 \\ -\cos\theta\frac{\sin n\theta}{4n\sin\theta} - \frac{\cos n\theta}{4}, & n > 0 \end{cases} \tag{70}$$

It can be easily verified that the operator $\tilde{W}_0$ defined by

$$\tilde{W}_0[\varphi](\theta) = -\frac{\tilde{\varphi}(\theta)}{4} - \frac{\cos\theta}{4}\tilde{C}[\tilde{\varphi}](\theta) + \frac{1-\ln 2}{4\pi}\int_0^\pi \tilde{\varphi}(\theta)d\theta, \tag{71}$$

reduces to the right-hand side of (70) when evaluated on the basis functions: $\tilde{J}_0[e_n] = \tilde{W}_0[e_n]$, for all $n \geq 0$ (the last term in equation (71) is obtained by collecting the zero-th order terms, and explicitly expressing the zero-th order coefficient of $\tilde{\varphi}$ as an integral). In view of Lemma 3 we see that the operator $\tilde{W}_0$ defines a bounded mapping from $H_e^s(2\pi)$ into $H_e^s(2\pi)$ for all $s > 0$. We conclude the equality of $\tilde{W}_0$ and $\tilde{J}_0$ from the continuity of $\tilde{J}_0$ established in Corollary 2, and the lemma follows. $\square$

From the relationship $\tilde{N}_0 = \tilde{J}_0\tilde{S}_0^{-1}$ we immediately obtain the following corollary.

Corollary 3 *The operator* $\tilde{N}_0 : \quad H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ *is continuous.*

Remark 2 *The decomposition* (71) *superficially resembles the classical closed-surface Calderón formula* (14)*, as it expresses the operator* $\tilde{W}_0 = \tilde{J}_0 = \tilde{N}_0\tilde{S}_0$ *as the sum of* $-I/4$ *and an additional operator. As shown in Section 3.4 below, however, the operator* $\tilde{C} : H_e^s(2\pi) \to H_e^s(2\pi)$ *which appears in* (71) *is not compact—and thus the Fredholm theory cannot be applied to establish the continuous invertibility of* $\tilde{J}_0 = \tilde{W}_0$ *merely on the basis of the decomposition* (71)*. The results of Section 3.3 nevertheless do establish that the operator* $\tilde{J}_0 = \tilde{N}_0\tilde{S}_0$ *is bicontinuous. Section* (3.4) *then provides a description of the spectrum of* $\tilde{J}_0$*, and, in preparation for the proof of Theorem 1, Section 3.5 extends all of these results to the operator* $\tilde{J}_0^\tau$*.*

3.3 Invertibility of $\tilde{J}_0$

We now proceed to show that the continuous operator $\tilde{J}_0 : H_e^s(2\pi) \to H_e^s(2\pi)$ admits a (bounded) inverse. Noting that the decomposition (45) is not directly invertible on a term by term basis ($\tilde{T}_0$ and $\tilde{D}_0$ are not invertible) we first state and proof two lemmas concerning the mapping properties of the operators $\tilde{T}_0$ and $\tilde{D}_0$.

Lemma 5 *The operators $\tilde{C}$ and $\tilde{T}_0$ satisfy*

$$\tilde{T}_0\tilde{C}[e_n] = \begin{cases} 0, & n = 0 \\ e_n, & n > 0. \end{cases} \tag{72}$$

and

$$\tilde{C}\tilde{T}_0[\tilde{\varphi}] = \tilde{\varphi} \quad \textit{for all} \quad \tilde{\varphi} \in H_e^s(2\pi). \tag{73}$$

Proof. In view of (47) and (62) we clearly obtain equation (72), while equation (73) follows immediately from (61). □

Lemma 6 *For all $s \geq 2$ the operator $\tilde{D}_0$ can be expressed in the form*

$$\tilde{D}_0 = -\frac{1}{4}\tilde{C}\left(\tilde{S}_0^{-1}\right)^2. \tag{74}$$

Proof. In view of (46) we have

$$\tilde{D}_0[e_n](\theta) = \begin{cases} 0, & n = 0 \\ -n\frac{\sin n\theta}{\sin\theta}, & n \geq 1. \end{cases} \tag{75}$$

In view of the continuity of the various operators involved in relevant Sobolev spaces (Lemmas 1, 2 and 3) and the density of the basis $\{e_n\}$ in $H_e^s(2\pi)$, equation (74) follows from (41) and (62). □

Corollary 4 *For all $s > 0$, the operator $\tilde{D}_0$ defines a bounded mapping from $H_e^{s+2}(2\pi)$ into $H_e^s(2\pi)$.*

Lemma 7 *For each integer $n \geq 2$ we have*

$$\tilde{C}\tilde{S}_0\tilde{T}_0[e_n](\theta) = \frac{\cos\theta}{2}\tilde{C}\left[\frac{n}{1-n^2}\ e_n\right](\theta) - \frac{n\cos n\theta}{2(1-n^2)}. \tag{76}$$

Proof. From the easily established identity

$$\tilde{S}_0\tilde{T}_0[e_n] = \frac{1}{4}(e_{n+1} - e_{n-1}) \quad (n \geq 1),$$

using equation (62) we obtain the relation

$$\tilde{C}\tilde{S}_0\tilde{T}_0[e_n] = \frac{1}{2}\left[\frac{\sin n\theta}{\sin\theta}\frac{\cos\theta}{1-n^2} - \frac{n\cos n\theta}{1-n^2}\right] \quad (n \geq 2) \tag{77}$$

which, via an additional application of equation (62) yields the desired equation (76). □

Corollary 5 *The composition $\tilde{C}\tilde{S}_0\tilde{T}_0$ which, in view of Lemma 1, Lemma 3 and equation (47), defines a continuous operator from $H_e^s(2\pi)$ to $H_e^s(2\pi)$ for $s \geq 1$, can in fact be extended in a unique fashion to an operator defined on $H_e^s(2\pi)$ for each $s \geq 0$. For all $s \geq 0$, further, these extended maps enjoy additional regularity: they can be viewed as continuous operators from $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$.*

Proof. The proof follows by consideration of equation (77). A cancellation of the form $n\sin(n+1)\theta - n\sin(n+1)\theta = 0$, which occurs in the process of evaluation of the right hand side of equation (77), underlies the additional regularity of the mapping $\tilde{C}\tilde{S}_0\tilde{T}_0$. □

We can now obtain the inverse of the operator $\tilde{J}_0$.

Lemma 8 *The continuous operator* $\tilde{J}_0 : H_e^s(2\pi) \to H_e^s(2\pi)$ *($s \geq 0$) which, according to equations (45) and (60) is given by*

$$\tilde{J}_0 = \tilde{D}_0 \tilde{S}_0 \tilde{T}_0 \tilde{S}_0, \tag{78}$$

is bijective, with (continuous) inverse $\tilde{J}_0^{-1} : H_e^s(2\pi) \to H_e^s(2\pi)$ *given by*

$$\tilde{J}_0^{-1} = -4\tilde{S}_0^{-1} \tilde{C} \tilde{S}_0 \tilde{T}_0 \tag{79}$$

for $s \geq 2$*, and given by the unique continuous extension of the right hand side of this equation for* $2 > s \geq 0$.

Proof.
Since the rightmost factor $\tilde{S}_0$ in equation (78) is a diagonal operator, we consider the next operator from the right in this product, namely, $\tilde{T}_0$, which in view of equations (72) and (73), admits $\tilde{C}$ as a "partial" inverse. Since the next factor $\tilde{S}_0$ from the right is, once again, a diagonal operator, we consider next the leftmost factor in equation (78): the operator $\tilde{D}_0$, a decomposition of which was provided in equation (74). In sum, to obtain the inverse of $\tilde{J}_0$ we proceed as follows: multiplying $\tilde{J}_0$ on the right by $\tilde{S}_0^{-1}\tilde{C}$ we obtain an operator that maps e_0 to 0 and e_n to $\tilde{D}_0\tilde{S}_0[e_n]$. Thus, considering (73) and (74), we further multiply on the right by $-4\tilde{S}_0\tilde{T}_0$ and we obtain the operator

$$-4\tilde{J}_0 \tilde{S}_0^{-1} \tilde{C} \tilde{S}_0 \tilde{T}_0 \tag{80}$$

which, in view of the fact that the image of $\tilde{S}_0\tilde{T}_0$ is orthogonal to e_0 (as it follows easily from equations (41) and (47)) maps e_n to $-4\tilde{D}_0 \left(\tilde{S}_0\right)^2 \tilde{T}_0[e_n]$ for all $n \geq 0$. But, in view of (74), this quantity equals $\tilde{C}\tilde{T}_0[e_n]$ which, according to (73), equals e_n. In other words, the operator (80), which is a continuous operator from $H_e^s(2\pi)$ to $H_e^{s-1}(2\pi)$ ($s \geq 1$), maps e_n to e_n for $n = 0, 1, 2 \ldots$— and, thus,

$$\tilde{I}_0 = -4\tilde{S}_0^{-1} \tilde{C} \tilde{S}_0 \tilde{T}_0 \tag{81}$$

is a right inverse of $\tilde{J}_0$, that is

$$\tilde{J}_0 \tilde{I}_0 = I, \tag{82}$$

at least for $s \geq 1$.

Conversely, since in view of equations (78) and (74) $\tilde{J}_0$ can be expressed, for $s \geq 1$, in the form

$$\tilde{J}_0 = -\frac{1}{4} \tilde{C} \tilde{S}_0^{-1} \tilde{T}_0 \tilde{S}_0, \tag{83}$$

for $s \geq 2$ we have

$$\tilde{I}_0 \tilde{J}_0 = \tilde{S}_0^{-1} \tilde{C} \tilde{S}_0 \tilde{T}_0 \tilde{C} \tilde{S}_0^{-1} \tilde{T}_0 \tilde{S}_0. \tag{84}$$

Now, as noted above, the image of $\tilde{T}_0$ is orthogonal to e_0, and thus, since $\tilde{S}_0$ is a diagonal operator, the same is true of the operator $\tilde{S}_0^{-1}\tilde{T}_0\tilde{S}_0$. Equation (72) can therefore be used directly to obtain

$$\tilde{T}_0 \tilde{C} \tilde{S}_0^{-1} \tilde{T}_0 \tilde{S}_0[e_n] = \tilde{S}_0^{-1} \tilde{T}_0 \tilde{S}_0[e_n], \quad \text{for all} \quad n \geq 0. \tag{85}$$

Clearly then, equation (84) can be reduced to

$$\tilde{I}_0 \tilde{J}_0 = \tilde{S}_0^{-1} \tilde{C} \tilde{T}_0 \tilde{S}_0, \tag{86}$$

and making use of (73), we finally obtain

$$\tilde{I}_0 \tilde{J}_0 = I, \tag{87}$$

as desired, thus establishing the invertibility of $\tilde{J}_0$ at least for $s \geq 2$. The boundedness of $\tilde{I}_0 = \tilde{J}_0^{-1}$ for $s \geq 2$ follows in view of Remark 1 (continuity of inverses of continuous linear maps) or, otherwise, directly from equation (79), Corollary 5 and Lemma 1. To treat the case $2 > s \geq 0$, finally, we note that by Corollary 5 and Lemma 1 $\tilde{I}_0$ can be extended in a unique fashion as a continuous mapping from $H_e^s(2\pi)$ to $H_e^s(2\pi)$ for all $s \geq 0$, and that by Lemma 4 $\tilde{J}_0$ is continuous mapping from $H_e^s(2\pi)$ to $H_e^s(2\pi)$ for all $s \geq 0$. The $s \geq 2$ relations (82) and (87) thus extend to all $s \geq 0$ by density of $H_e^2(2\pi)$ in $H_e^s(2\pi)$ $(2 > s \geq 0)$, and the proof is thus complete. □

Corollary 6 *For all $s \geq 0$, the operator $\tilde{N}_0 = \tilde{J}_0 \tilde{S}_0^{-1}$ defines a bicontinuous mapping from $H_e^{s+1}(2\pi)$ to $H_e^s(2\pi)$.*

Proof. This follows directly from equation (39), equation (60), and Lemmas 4 and 8. □

3.4 Point Spectrum of $\tilde{J}_0$

Having established boundedness and invertibility, we conclude our study of the operator $\tilde{J}_0$ by computing its eigenvalues.

Lemma 9 *For any $s > 0$, the point spectrum σ_s of $\tilde{J}_0 : H_e^s(2\pi) \to H_e^s(2\pi)$ can be expressed as the union*

$$\sigma_s = \Lambda_s \cup \Lambda_\infty, \tag{88}$$

where Λ_∞ is the discrete set

$$\Lambda_\infty = \left\{\lambda_n : n = 0, 1, \ldots, \infty\right\}, \ \lambda_n = \begin{cases} -\frac{\ln 2}{4}, & n = 0 \\ -\frac{1}{4} - \frac{1}{4n}, & n > 0, \end{cases} \tag{89}$$

and where Λ_s is the open bounded set

$$\Lambda_s = \left\{ \lambda = (\lambda_x + i\lambda_y) \in \mathbb{C} \ : \ 4s + 2 < \frac{-\left(\lambda_x + \frac{1}{4}\right)}{(\lambda_x + \frac{1}{4})^2 + \lambda_y^2} \right\}. \tag{90}$$

Proof. We start by re-expressing equation (70) as

$$\tilde{J}_0[e_n](\theta) = \begin{cases} -\frac{\ln 2}{4} & n = 0 \\ -\frac{\sin(n+1)\theta}{4n \sin\theta} + \frac{\cos n\theta}{4n} - \frac{\cos n\theta}{4}, & n > 0. \end{cases} \tag{91}$$

Then, making use again of (49) we obtain

$$\tilde{J}_0[e_n] = \begin{cases} \lambda_n e_n - \frac{1}{2n} \sum\limits_{k=0}^{p-1} (1 - \frac{\delta_{0k}}{2}) e_{2k}, & n = 2p, \ p \geq 0 \\ \lambda_n e_n - \frac{1}{2n} \sum\limits_{k=0}^{p-1} e_{2k+1}, & n = 2p + 1, \ p \geq 0 \end{cases}, \tag{92}$$

where the diagonal elements λ_n are defined in equation (89). Clearly, $\tilde{J}_0$ takes the form of an upper-triangular (infinite) matrix whose diagonal terms λ_n define eigenvalues associated with eigenvectors v_n, each one of which can be expressed in terms of a finite linear combination of the first n basis functions: $v_n = \sum_{k=0}^{n} c_k^n e_k$. In particular, for all $n \in \mathbb{N}$, $v_n \in H_e^s[0, 2\pi]$ for all $s > 0$. This shows that the set Λ_∞ of diagonal elements defined in equation (89) is indeed contained in σ_s for all $s > 0$.

As is well known, an upper triangular operator in an infinite-dimensional space can have eigenvalues beyond those represented by diagonal elements. As shown in [8, Th. 2], for instance, the point spectrum of the upper-triangular bounded operator

$$C^*[a](n) = \sum_{k=n}^{\infty} \frac{a_k}{k+1}, \tag{93}$$

(the adjoint of the discrete Cesàro operator C) is the open disc $|\lambda - 1| < 1$. A similar situation arises for our operator $\tilde{J}_0$.

To obtain the full point spectrum of the operator $\tilde{J}_0$ let $\lambda \in \mathbb{C}$ and $f = \sum_{k=0}^{\infty} f_n e_n$ be such that $\tilde{J}_0[f] = \lambda f$. It follows from (92) that the coefficients f_n satisfy the relation

$$(-\frac{1}{4} - \frac{1}{4n})f_n - \frac{1}{2}\sum_{k=1}^{\infty} \frac{f_{n+2k}}{n+2k} = \lambda f_n \quad , \quad n \geq 1, \tag{94}$$

along with

$$(-\frac{\ln 2}{4})f_0 - \frac{1}{4}\sum_{k=1}^{\infty} \frac{f_{2k}}{2k} = \lambda f_0 \quad , \quad n = 0. \tag{95}$$

Equation (94) is equivalent to

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{f_{n+2k}}{n+2k} = f_n(-\frac{1}{4n} - \frac{1}{4} - \lambda), \quad n \geq 1, \tag{96}$$

which, by subtraction, gives

$$\frac{1}{2}\frac{f_{n+2}}{n+2} = f_n(-\frac{1}{4n} - \frac{1}{4} - \lambda) - f_{n+2}(-\frac{1}{4(n+2)} - \frac{1}{4} - \lambda), \quad n \geq 1. \tag{97}$$

Therefore, the coefficients of f must satisfy

$$\begin{cases} f_{n+2} = & f_n \left(\frac{\frac{z}{2} + \frac{1}{n}}{\frac{z}{2} - \frac{1}{(n+2)}} \right), \qquad n \geq 1 \\ \frac{1}{4}\sum_{k=1}^{\infty} \frac{f_{2k}}{2k} = & f_0(-\frac{\ln 2}{4} - \lambda), \qquad n = 0. \end{cases} \tag{98}$$

where, in order to simplify the notations, we write

$$z = 8\lambda + 2. \tag{99}$$

It is clear from equation (98) that the zero-th coefficient is determined by the coefficients of even positive orders, and that the sequence f_n for $n \geq 1$ is entirely determined by f_1 and f_2.

Clearly, there are no elements of the point spectrum for which $Re(z) \geq 0$, since for such values of z the resulting sequence f_n is not square summable (that is, $\sum |f_n|^2 = \infty$). Note that the set of vectors $\{v_n\}$ associated with the discrete eigenvalues $\lambda_n = -\frac{1}{4} - \frac{1}{4n}$, in turn, are recovered by setting $z = -\frac{2}{n}$. To determine all of the elements of the point spectrum with $Re(z) < 0$ we study separately the odd and even terms in the sequence (98). We start with the sequence $q_n = f_{2n}$, which satisfies the recurrence relationship

$$q_{n+1} = q_n \left(\frac{z + \frac{1}{n}}{z - \frac{1}{n+1}} \right), \quad n \geq 1. \tag{100}$$

Let $z = -x + iy$ with $x > 0$, and assume without loss of generality, that $q_1 = 1$. Then

$$q_n = \left(\frac{z-1}{z-\frac{1}{n}}\right) \prod_{k=1}^{n-1} \left(\frac{z+\frac{1}{k}}{z-\frac{1}{k}}\right), \quad n \geq 1, \tag{101}$$

and it follows that

$$\begin{aligned} \ln|q_n| &= \ln\left|\frac{z-1}{z-\frac{1}{n}}\right| + \frac{1}{2}\sum_{k=1}^{n-1} \ln\left(\frac{(x-\frac{1}{k})^2+y^2}{(x+\frac{1}{k})^2+y^2}\right) \\ &= \ln\left|\frac{z-1}{z-\frac{1}{n}}\right| + \frac{1}{2}\sum_{k=1}^{n-1} \ln\left(\frac{1-r(x,y,k))}{1+r(x,y,k)}\right) \end{aligned} \tag{102}$$

where

$$r(x,y,k) = \frac{2x}{k(x^2+y^2+\frac{1}{k^2})}. \tag{103}$$

For large k, we have

$$\ln\left(\frac{1-r(x,y,k)}{1+r(x,y,k)}\right) = -\frac{4x}{k(x^2+y^2)} + O(\frac{1}{k^3}),\ k \to \infty, \tag{104}$$

and thus

$$\ln|q_n| = -\frac{2x}{x^2+y^2}\ln n + M + O(\frac{1}{n}), \tag{105}$$

where M is a constant. The absolute value of q_n is thus asymptotically given by

$$|q_n| = O\left(\frac{1}{n^{\frac{2x}{x^2+y^2}}}\right) \tag{106}$$

as $n \to \infty$. It follows that, for any $s > 0$, the set of points (x, y) in the half plane such that the sequence $\sum n^{2s}|q_n|^2 < \infty$ is exactly defined by the equation

$$2s - \frac{4x}{x^2+y^2} < -1. \tag{107}$$

The analysis for the odd-term sequence $p_n = f_{2n+1}$ can be carried out similarly, since

$$p_{n+1} = p_n \left(\frac{z+\frac{1}{n+\frac{1}{2}}}{z-\frac{1}{n+1+\frac{1}{2}}}\right), \tag{108}$$

which essentially amounts to replacing k by $k+\frac{1}{2}$ in equations (102) and (103). The convergence condition (107) thus applies to p_n as well, and it follows, in view of equation (99), that the set Λ_s defined by (90) contains all the eigenvalues of $\tilde{J}_0$ not contained in Λ_∞. $\square$

Corollary 7 *The operator* $\tilde{C} : H_e^s(2\pi) \to H_e^s(2\pi)$ *is not compact.*

Proof. This follows from the decomposition (71) of $\tilde{J}_0$ and the fact that $\tilde{J}_0$ admits a spectrum that is not discrete. $\square$

Remark 3 *Using polar coordinates* (r, θ) *around the point* $(-\frac{1}{4}, 0)$ *it is easy to check that*

$$\Lambda_s = \left\{ (\lambda_x + i\lambda_y) \in \mathbb{C} : \lambda_x + \frac{1}{4} = r\cos\theta,\ \lambda_y = r\sin\theta,\ 0 < r < -\frac{\cos\theta}{4s+2},\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \right\}. \quad (109)$$

Clearly then, for $s > s'$, $\Lambda_s \subsetneq \Lambda_{s'}$, *and we have* $\bigcap_{s>0} \Lambda_s = \varnothing$, *while the intersection of the closures is given by* $\bigcap_{s>0} \bar{\Lambda}_s = \{-\frac{1}{4}\}$. *Also, for all* $s > 0$, $\operatorname{dist}(\sigma_s, 0) = -\frac{1}{4}$, *and* $\max_{\lambda \in \sigma_s} |\lambda| \leq \frac{3}{4}$. *It therefore follows that* σ_s *is bounded away from the zero and infinity. In view of Theorem 1 and Section 3.5, this is a fact of great significance in connection with the numerical solution of equations* (26) *and* (27) *by means of Krylov-subspace iterative linear-algebra techniques; see [12] for details.*

3.5 The operator $\tilde{J}_0^\tau$

In our proof of Theorem 1 we need to consider not $\tilde{J}_0$ but a closely related operator, namely

$$\tilde{J}_0^\tau = \tilde{N}_0^\tau \tilde{S}_0^\tau \quad (110)$$

where defining (in a manner consistent with equation (117) below) $\tilde{Z}_0[\gamma](\theta) = \gamma(\theta)\tau(\cos\theta)$, we have set

$$\tilde{S}_0^\tau[\gamma] = \tilde{S}_0 \tilde{Z}_0[\gamma], \quad (111)$$

and

$$\tilde{N}_0^\tau[\gamma] = \tilde{Z}_0^{-1} \tilde{N}_0[\gamma]. \quad (112)$$

It is easy to generalize equation (39), Corollary 6 and Lemmas 4 through 9 to needed corresponding results for $\tilde{S}_0^\tau$, $\tilde{N}_0^\tau$ and $\tilde{J}_0^\tau$; these are given in the following Theorem.

Theorem 2 *Let* $s \geq 0$. *Then,*

(i) The operator $\tilde{S}_0^\tau : H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ *is bicontinuous.*

(ii) The operator $\tilde{N}_0^\tau : H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ *is bicontinuous.*

(iii) The operator $\tilde{J}_0^\tau : H_e^s(2\pi) \to H_e^s(2\pi)$ *is bicontinuous.*

(iv) The point spectrum of $\tilde{J}_0^\tau : H_e^s(2\pi) \to H_e^s(2\pi)$ *is equal to the point spectrum* σ_s *of* $\tilde{J}_0$.

Proof. In view of (111), (112), the ensuing relation

$$\tilde{J}_0^\tau = \tilde{Z}_0^{-1} \tilde{J}_0 \tilde{Z}_0, \quad (113)$$

and the fact that τ is smooth and non-vanishing, the proof of points *(i)*, *(ii)* and *(iii)* is immediate. Equation (113) also shows that (λ, v) is an eigenvalue-eigenvector pair for $\tilde{J}_0$ if and only if $(\lambda, \tilde{Z}_0^{-1}[v])$ is an eigenvalue-eigenvector pair for $\tilde{J}_0^\tau$, and point *(iv)* follows as well. □

4 General Properties of the Operators $\tilde{S}$ and $\tilde{N}$

The proof of Theorem 1 results from a perturbation argument involving Theorem 2 and the results established in this section on the regularity and invertibility of the operators $\tilde{S}$ and $\tilde{N}$ defined by equations (28) and (29).

4.1 Bicontinuity of the operator $\tilde{S}$

We seek to show that for all $s \geq 0$ the operator $\tilde{S}$ defined in equation (28) is a bicontinuous mapping between $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$. This is done in Lemmas 10 and 12 below.

Lemma 10 *Let $s \geq 0$. Then $\tilde{S}$ defines a bounded mapping from $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$. Further, the difference $\tilde{S} - \tilde{S}_0^\tau$ (see equation (111)) defines a continuous mapping from $H_e^s(2\pi)$ into $H_e^{s+3}(2\pi)$.*

Proof. In view of equation (4) and the expression

$$H_0^1(z) = \frac{2i}{\pi} J_0(z) \ln(z) + R(z)$$

for the Hankel function in terms of the Bessel function $J_0(z)$, the logarithmic function and a certain entire function R, the kernel of the operator S_ω (equation (22)) can be cast in the form

$$G_k(\mathbf{r}(t), \mathbf{r}(t')) = A_1(t,t') \ln|t - t'| + A_2(t,t'), \tag{114}$$

where $A_1(t,t')$ and $A_2(t,t')$ are smooth functions. Further, since $J_0(z)$ is given by a series in powers of z^2, it follows that for all $m \in \mathbb{N}$, the function A_1 can be expressed in the form

$$A_1(t,t') = -\frac{1}{2\pi} + \sum_{n=2}^{m+3} a_n(t)(t'-t)^n + (t-t')^{m+4} \Lambda_{m+3}(t,t'),$$

where $\Lambda_{m+3}(t,t')$ is a smooth function of t and t'. The operator $\tilde{S}$ in equation (28) can thus be expressed in the form

$$\begin{aligned} \tilde{S}[\tilde{\varphi}](\theta) = \tilde{S}_0^\tau[\tilde{\varphi}](\theta) + \sum_{n=2}^{m+3} a_n(\cos\theta) \int_0^\pi (\cos\theta' - \cos\theta)^n \ln|\cos\theta - \cos\theta'| \tilde{\varphi}(\theta') \tau(\cos\theta') d\theta' \\ + \int_0^\pi A_3(\cos\theta, \cos\theta') \tilde{\varphi}(\theta') \tau(\cos\theta') d\theta', \end{aligned} \tag{115}$$

where $A_3(\cos\theta, \cos\theta')$, which contains a logarithmic factor, belongs to $C^{m+3}([0,2\pi] \times [0,2\pi])$.

Clearly, for $n \geq 2$, the second derivative $d^2/d\theta^2$ of the product $(\cos\theta' - \cos\theta)^n \ln|\cos\theta - \cos\theta'|$ can be expressed as a product $P_1(\cos\theta, \cos\theta') \ln|\cos\theta - \cos\theta'| + P_2(\cos\theta, \cos\theta')$ where $P_1(t,t')$ and $P_2(t,t')$ are polynomials. Collecting terms with the common factor $\cos^\ell \theta'$ we then obtain

$$\frac{d^2}{d\theta^2} \left(\tilde{S} - \tilde{S}_0^\tau \right) [\tilde{\varphi}](\theta) = \sum_{\ell=0}^{m+1} b_\ell(\cos\theta) \tilde{S}_0 \tilde{Z}_\ell[\tilde{\varphi}](\theta) + \int_0^\pi A_4(\cos\theta, \cos\theta') \tilde{\varphi}(\theta') \tau(\cos\theta') d\theta', \tag{116}$$

where $b_\ell(\cos\theta)$ is an even smooth function, where the operator $\tilde{Z}_\ell : H_e^s(2\pi) \to H_e^s(2\pi)$ $(s \in \mathbb{R})$ is given by

$$\tilde{Z}_\ell[\gamma](\theta') = \cos^\ell \theta' \, \tau(\cos\theta') \, \gamma(\theta'), \tag{117}$$

and where $A_4(\cos\theta, \cos\theta') \in C^{m+1}([0,2\pi] \times [0,2\pi])$. Now, in view of equation (39), the first term on the right-hand-side of equation (116) defines a bounded operator from $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$. On the other hand, the derivatives of orders $k \leq (m+1)$ of the second term on the right-hand-side of (116), all reduce to integral operators with bounded kernels, and thus map $L^2[0,2\pi]$ continuously into $L^2[0,2\pi]$. It follows that the second term itself maps continuously $H_e^0(2\pi)$ (and hence $H_e^m(2\pi)$)

into $H_e^{m+1}(2\pi)$, and the lemma follows for integer values $s = m$. The extension for real values $s > 0$ follows directly by interpolation [22, Theorem 8.13]. $\square$

The following lemma and its corollary provide a direct link, needed for our proof of Lemma 12, between the spaces $H_e^s(2\pi)$ under consideration here and the original space $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ appearing in equations (10).

Lemma 11 *Let $s > 0$, and assume $\tilde{\varphi} \in H_e^s(2\pi)$. Then the function*

$$w(\xi) = \frac{1}{\pi} \int_0^\pi \tilde{\varphi}(\theta) e^{-i\xi\cos\theta} d\theta. \tag{118}$$

satisfies

$$\int_{\mathbb{R}} \frac{|w(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi < \infty. \tag{119}$$

Proof. Using the $L^2[0,\pi]$-convergent cosine expansion

$$\tilde{\varphi}(\theta) = \sum_{n=0}^{\infty} a_n \cos\theta \tag{120}$$

we obtain

$$w(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{\pi} \int_0^\pi \cos n\theta e^{-i\xi\cos\theta} d\theta. \tag{121}$$

Since

$$\int_0^\pi \cos n\theta e^{-i\xi\cos\theta} d\theta = \frac{1}{2}\int_{-\pi}^{\pi} e^{in\theta} e^{-i\xi\cos\theta} d\theta = \frac{1}{2} e^{\frac{in\pi}{2}} \int_{-\pi}^{\pi} e^{-in\theta} e^{-i\xi\sin\theta} d\theta = \pi i^n J_n(-\xi), \tag{122}$$

(where, denoting by $J_n(\xi)$ the Bessel function of order n, the last identity follows from [19, 8.411 p. 902]), we see that equation (121) can be re-expressed in the form

$$w(\xi) = \sum_{n=0}^{\infty} i^n a_n J_n(-\xi) = \sum_{n=0}^{\infty} \left(\sqrt{1+n^{2s}}\; i^n a_n\right) \left(\frac{J_n(-\xi)}{\sqrt{1+n^{2s}}}\right). \tag{123}$$

In view of the Cauchy-Schwartz inequality we thus obtain

$$|w(\xi)|^2 \leq \left(\sum_{n=0}^{\infty} (1+n^{2s})|a_n|^2\right) \left(\sum_{n=0}^{\infty} \frac{|J_n(\xi)|^2}{1+n^{2s}}\right) \leq \left(\sum_{n=1}^{\infty} \frac{|J_n(\xi)|^2}{n^{2s}} + |J_0(\xi)|^2\right) \|\tilde{\varphi}\|_s^2. \tag{124}$$

Since $0 \leq |\xi|/(1+|\xi|^2)^{1/2} \leq 1$, it follows that

$$\begin{aligned} \int_{\mathbb{R}} \frac{|w(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi &\leq \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^{2s}} \int_{\mathbb{R}} \frac{|J_n(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi\right) + \int_{\mathbb{R}} \frac{|J_0(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi\right) \|\tilde{\varphi}\|_s^2 \\ &\leq \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^{2s}} \int_{\mathbb{R}} \frac{|J_n(\xi)|^2}{|\xi|} d\xi\right) + \int_{\mathbb{R}} \frac{|J_0(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi\right) \|\tilde{\varphi}\|_s^2. \end{aligned} \tag{125}$$

Further, in view of [19, 6.574, eq 2.], the integral involving J_n can be computed exactly for $n \geq 1$:

$$\int_{\mathbb{R}} \frac{|J_n(\xi)|^2}{|\xi|} d\xi = \frac{1}{n}. \tag{126}$$

It thus follows that

$$\int_{\mathbb{R}} \frac{|w(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi \le C_s \|\tilde{\varphi}\|_s^2 < \infty \tag{127}$$

where

$$C_s = \sum_{n=1}^{\infty} \frac{1}{n^{1+2s}} + \int_{\mathbb{R}} \frac{|J_0(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi. \tag{128}$$

□

Corollary 8 *Let $s > 0$, $\tilde{\varphi} \in H_e^s(2\pi)$, $\varphi(t) = \tilde{\varphi}(\arccos(t))$, $\varphi : [-1,1] \to \mathbb{C}$, $\alpha(\mathbf{p}) = \varphi(\mathbf{r^{-1}}(\mathbf{p}))$ and $W(\mathbf{p}) = \omega(\mathbf{r^{-1}}(\mathbf{p}))$. Then, the function $F = \frac{\alpha}{W}$ is an element of $\tilde{H}^{-\frac{1}{2}}(\Gamma)$.*

Proof. It suffices to take show that $f = \varphi/\omega \in \tilde{H}^{-\frac{1}{2}}[-1,1]$ for the case $\Gamma = [-1,1]$. Extending f by 0 outside the interval $[-1,1]$, the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt = \int_{-1}^{1} \frac{\varphi(t) e^{-i\xi t}}{\omega(t)} dt = \int_0^{\pi} \tilde{\varphi}(\theta) e^{-i\xi\cos\theta} d\theta, \tag{129}$$

since $\omega(t) = \sqrt{1-t^2}$ in the present case. The Corollary now follows from Lemma 11. □

Lemma 12 *For all $s > 0$ the operator $\tilde{S} : H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ is invertible, and the inverse $\tilde{S}^{-1} : H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ is a bounded operator.*

Proof. Let $s > 0$ be given. From Lemma 1 we know $\tilde{S}_0 : H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ is a continuously invertible operator. The same clearly holds for $\tilde{S}_0^\tau$ as well, and we may write

$$\tilde{S} = \tilde{S}_0^\tau \left(I + \left(\tilde{S}_0^\tau\right)^{-1} (\tilde{S} - \tilde{S}_0^\tau) \right). \tag{130}$$

It follows from Lemma 10 that the operator $(\tilde{S}_0^\tau)^{-1}(\tilde{S} - \tilde{S}_0^\tau)$ is bounded from $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$, and therefore, in view of the Sobolev embedding theorem it defines a compact mapping from $H_e^s(2\pi)$ into itself. Further, in view of Corollary 8 and the injectivity of the mapping (10) it follows that the operator $\tilde{S} : H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ is injective, and therefore, so is

$$\left(\tilde{S}_0^\tau\right)^{-1} \tilde{S} = I + \left(\tilde{S}_0^\tau\right)^{-1} (\tilde{S} - \tilde{S}_0^\tau) : H_e^s(2\pi) \to H_e^s(2\pi). \tag{131}$$

A direct application of the Fredholm theory thus shows that the operator (131) is continuously invertible, and the lemma follows. □

4.2 Bicontinuity of the operator $\tilde{N}$

To study the mapping properties of the operator $\tilde{N}$ we rely on Lemma 13 below where, as in [28], the operator $\tilde{N}$ is re-cast in terms of an expression which involves tangential differential operators (cf. also [15, Th. 2.23] for the corresponding result for closed surfaces). The needed relationships between normal vectors, tangent vectors and parametrizations used are laid down in the following definition.

Definition 4 *For a given (continuous) selection of the normal vector $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{r})$ on Γ, the tangent vector $\boldsymbol{t}(\boldsymbol{r})$ is the unit vector that results from a 90° clockwise rotation of $\boldsymbol{n}(\boldsymbol{r})$. Throughout this paper it is further assumed that the parametrization $\boldsymbol{r} = \boldsymbol{r}(t)$ of the curve Γ has been selected in such a way that*

$$\frac{d\boldsymbol{r}}{dt}(t) = \left|\frac{d\boldsymbol{r}}{dt}\right| \boldsymbol{t}(\boldsymbol{r}(t)). \tag{132}$$

Lemma 13 *For $\varphi \in C^\infty(\Gamma)$, and for $t \in (-1,1)$, the quantity $N_\omega[\varphi](t)$ defined by equation (23) can be expressed in the form*

$$N_\omega[\varphi](t) = N_\omega^g[\varphi](t) + N_\omega^{pv}[\varphi](t) \tag{133}$$

where

$$N_\omega^g[\varphi](t) = k^2 \int_{-1}^{1} G_k(\mathbf{r}(t), \mathbf{r}(t'))\, \varphi(t')\, \tau(t')\, \sqrt{1-t'^2}\, \mathbf{n}_t \cdot \mathbf{n}_{t'}\, dt', \tag{134}$$

and where

$$N_\omega^{pv}[\varphi](t) = \frac{1}{\tau(t)} \frac{d}{dt} \left(\int_{-1}^{1} G_k(\mathbf{r}(t), \mathbf{r}(t'))\, \frac{d}{dt'} \left(\varphi(t')\, \sqrt{1-t'^2} \right) dt' \right). \tag{135}$$

Proof. See Appendix A, cf. [15, 22, 28]. $\square$

In order to continue with our treatment of the operator $\tilde{N}$ we note that, using the changes of variables $t = \cos\theta$ and $t' = \cos\theta'$ in equations (134) and (135) together with the notation (31), for $\varphi \in C^\infty(\Gamma)$ and for $\theta \in (0, \pi)$ we obtain

$$\tilde{N}[\tilde{\varphi}] = \tilde{N}^g[\tilde{\varphi}] + \tilde{N}^{pv}[\tilde{\varphi}], \tag{136}$$

where

$$\tilde{N}^g[\tilde{\varphi}](\theta) = k^2 \int_0^\pi G_k(\mathbf{r}(\cos\theta), \mathbf{r}(\cos\theta'))\, \tilde{\varphi}(\theta')\, \tau(\cos\theta')\, \sin^2\theta'\, \mathbf{n}_\theta \cdot \mathbf{n}_{\theta'}\, d\theta', \tag{137}$$

and where, taking into account equations (46) and (47),

$$\tilde{N}^{pv}[\tilde{\varphi}](\theta) = \frac{1}{\tau(\cos\theta)} \left(\tilde{D}_0 \tilde{S} \tilde{T}_0^\tau \right) [\tilde{\varphi}](\theta), \tag{138}$$

with

$$\tilde{T}_0^\tau[\tilde{\varphi}](\theta) = \frac{1}{\tau(\cos\theta)} T_0[\tilde{\varphi}](\theta). \tag{139}$$

Lemma 14 *Let $s \geq 0$. The operator $\tilde{N}^{pv}$ defines a bounded mapping from $H_e^{s+1}(2\pi)$ to $H_e^s(2\pi)$. Further, the difference $(\tilde{N}^{pv} - \tilde{N}_0^\tau)$ (see equation (112)) defines a bounded mapping from $H_e^{s+1}(2\pi)$ into $H_e^{s+1}(2\pi)$.*

Proof. Using (45), (112) and (138) we obtain

$$\tilde{N}^{pv}[\tilde{\varphi}] = \tilde{N}_0^\tau[\tilde{\varphi}] + \frac{1}{\tau(\cos\theta)} \tilde{D}_0 (\tilde{S} - \tilde{S}_0^\tau) \tilde{T}_0^\tau[\tilde{\varphi}]. \tag{140}$$

As shown in Theorem 2 the operator $\tilde{N}_0^\tau : H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ on the right-hand side of this equation is bounded. To establish the continuity of the second term on the right-hand side of equation (140) we first note that, in view of equation (47), the operator $\tilde{T}_0 : H_e^{s+1}(2\pi) \to H_e^s(2\pi)$

is bounded, and therefore, so is $\tilde{T}_0^\tau$. Further, as shown in Lemma 10, the operator $(\tilde{S} - \tilde{S}_0^\tau)$ maps continuously $H_e^s(2\pi)$ into $H_e^{s+3}(2\pi)$ so that, to complete the proof, it suffices to show that the operator $\tilde{D}_0$ maps continuously $H_e^{s+3}(2\pi)$ into $H_e^{s+1}(2\pi)$. But, for $\tilde{\psi} \in H_e^{s+3}(2\pi)$ $(s > 0)$ we can write

$$\tilde{D}_0[\tilde{\psi}](\theta) = \frac{1}{\sin\theta} \int_0^\theta \frac{d^2}{d\theta^2} \tilde{\psi}(u) du,$$

and since the zero-th order term in the cosine expansion of $\frac{d^2}{d\theta^2}\tilde{\psi}$ vanishes, in view of (62) we have

$$\tilde{D}_0[\tilde{\psi}] = \tilde{C}\left[\frac{d^2\tilde{\psi}}{d\theta^2}\right].$$

It therefore follows from Lemma 3 that the second term in (140) is a continuous map from $H_e^{s+1}(2\pi)$ into $H_e^s(2\pi)$, that is, $(\tilde{N}^{pv} - \tilde{N}_0^\tau)$, as claimed. $\square$

Corollary 9 *For all $s \geq 0$ the operator $\tilde{N}$ can be extended as a continuous linear map from $H_e^{s+1}(2\pi)$ to $H_e^s(2\pi)$. Further, the difference $\tilde{N} - \tilde{N}_0^\tau$ defines a continuous operator from $H_e^{s+1}(2\pi)$ to $H_e^{s+1}(2\pi)$.*

Proof. From equation (137) we see that $\tilde{N}^g$ has the same mapping properties as $\tilde{S}$ (Lemma 10), namely

$$\tilde{N}^g : H_e^{s+1}(2\pi) \to H_e^{s+2}(2\pi) \quad \text{is continuous.} \tag{141}$$

In view of Lemma 14 it therefore follows that the right hand side of equation (136),

$$\tilde{N}^g + \tilde{N}^{pv} : H_e^{s+1}(2\pi) \to H_e^s(2\pi), \tag{142}$$

is a bounded operator for all $s \geq 0$. Equation (136) was established for functions $\tilde{\varphi}$ of the form (31) with $\varphi \in C^\infty(\Gamma)$. But the set of such functions $\tilde{\varphi}$ is dense in $H_e^{s+1}(2\pi)$ for all $s > 0$—as can be seen by considering, e.g., that the Chebyshev polynomials span a dense set in $H^{s+1}[-1,1]$. It follows that $\tilde{N}$ can be uniquely extended to a continuous operator from $H_e^{s+1}(2\pi)$ to $H_e^s(2\pi)$, as claimed. Finally, $\tilde{N} - \tilde{N}_0^\tau = \tilde{N}^g + (\tilde{N}^{pv} - \tilde{N}_0^\tau)$ is continuous from $H_e^{s+1}(2\pi)$ into $H_e^{s+1}(2\pi)$, in view of equation (141) and Lemma 14.
$\square$

The following lemma establishes a link, needed for our proof of Lemma 16, between the domain of the unweighted hypersingular operator $\mathbf{N}$ considered in [34] (equation (11) above) and the corresponding possible domains of the weighted operator $\tilde{N}$ (equation (37)); cf. also Corollary 8 where the corresponding result for the domains of the operators $\mathbf{S}$ and $\tilde{S}$ is given.

Lemma 15 *Let $\tilde{\psi}$ belong to $H_e^{s+1}(2\pi)$ for $s > 0$, $\psi(t) = \tilde{\psi}(\arccos t)$, $\psi : [-1,1] \to \mathbb{C}$, $\beta(\mathbf{p}) = \psi\left(\mathbf{r^{-1}}(\mathbf{p})\right)$, $W(\mathbf{p}) = \omega\left(\boldsymbol{r}^{-1}(\mathbf{p})\right)$. Then the function $G = W\beta$ is an element of $\tilde{H}^{\frac{1}{2}}(\Gamma)$.*

Proof. It suffices to show that $g = \omega\psi \in \tilde{H}^{\frac{1}{2}}[-1,1]$ for the case $\Gamma = [-1,1]$. Extending g by 0 outside the interval $[-1,1]$, the Fourier transform of g is given by

$$\hat{g}(\xi) = \int_{-1}^1 \psi(t) e^{-i\xi t}\omega(t) dt = \int_0^\pi \psi(\cos\theta) e^{-i\xi\cos\theta} \sin^2\theta d\theta, \tag{143}$$

since $\omega(t) = \sqrt{1-t^2}$ in the present case. Integrating by parts we obtain

$$\hat{g}(\xi) = \frac{1}{i\xi}\int_0^\pi \frac{\partial}{\partial\theta}\left\{\psi(\cos\theta)\sin\theta\right\} e^{-i\xi\cos\theta} d\theta. \tag{144}$$

It is easy to check that $\frac{\partial}{\partial\theta}\{\psi(\cos\theta)\sin\theta\} = \frac{\partial}{\partial\theta}\{\tilde{\psi}(\theta)\sin\theta\}$ is an element of $H_e^s(2\pi)$ and, thus, in view of equation (144) together with Lemma 11 we obtain

$$\int_{\mathbb{R}} \frac{|\hat{g}(\xi)|^2\xi^2}{(1+\xi^2)^{\frac{1}{2}}} d\xi < \infty \tag{145}$$

It thus follows that the second term on the right-hand side of the identity

$$\int_{\mathbb{R}} |\hat{g}(\xi)|^2(1+|\xi|^2)^{\frac{1}{2}} d\xi = \int_{\mathbb{R}} \frac{|\hat{g}(\xi)|^2}{(1+\xi^2)^{\frac{1}{2}}} d\xi + \int_{\mathbb{R}} \frac{|\hat{g}(\xi)|^2\xi^2}{(1+\xi^2)^{\frac{1}{2}}} d\xi \tag{146}$$

is finite. The first term is also finite, as can be seen by applying Lemma 11 directly to equation (143). The function g thus belongs to $\tilde{H}^{\frac{1}{2}}[-1,1]$, and the proof is complete. □

Lemma 16 *For all $s > 0$ the operator $\tilde{N} : H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ is invertible, and the inverse $\tilde{N}^{-1} : H_e^s(2\pi) \to H_e^{s+1}(2\pi)$ is a bounded operator.*

Proof. In view of Theorem 2, the operator $\tilde{N}_0^\tau \ : \ H_e^{s+1}(2\pi) \to H_e^s(2\pi)$ is bicontinuous, and we may thus write

$$\tilde{N} = \tilde{N}_0^\tau \left(I + \left(\tilde{N}_0^\tau\right)^{-1} (\tilde{N} - \tilde{N}_0^\tau) \right). \tag{147}$$

Since, by Corollary 9, the difference $\tilde{N} - \tilde{N}_0^\tau$ defines a bounded mapping from $H_e^{s+1}(2\pi)$ into $H_e^{s+1}(2\pi)$, it follows that the operator $\left(\tilde{N}_0^\tau\right)^{-1} (\tilde{N} - \tilde{N}_0^\tau)$ is bounded from $H_e^{s+1}(2\pi)$ into $H_e^{s+2}(2\pi)$ and, in view of the Sobolev embedding theorem, it is also compact from $H_e^{s+1}(2\pi)$ into $H_e^{s+1}(2\pi)$. The Fredholm theory can thus be applied to the operator

$$I + \left(\tilde{N}_0^\tau\right)^{-1} (\tilde{N} - \tilde{N}_0^\tau). \tag{148}$$

This operator is also injective, in view of Lemma 15 and the bicontinuity of the map $\mathbf{N}$ in equation (11), and it is therefore invertible. The Lemma then follows from the bicontinuity of the operator of $\tilde{N}_0^\tau$. □

5 Generalized Calderón Formula: Proof of Theorem 1

Collecting results presented in previous sections we can now present a proof of Theorem 1.
Proof. The bicontinuity of the operators $\tilde{S}$, $\tilde{N}$ and $\tilde{N}\tilde{S}$ follow directly from Lemmas 10, 12, 16 and Corollary 9. To establish equation (35), on the other hand, we write

$$\tilde{N}\tilde{S} = \tilde{N}_0^\tau \tilde{S}_0^\tau + \tilde{K} = \tilde{J}_0^\tau + \tilde{K} \tag{149}$$

where as shown in Theorem 2, $\tilde{J}_0^\tau$ is bicontinuous, and where

$$\tilde{K} = \tilde{N}(\tilde{S} - \tilde{S}_0^\tau) + (\tilde{N} - \tilde{N}_0^\tau)\tilde{S}_0^\tau. \tag{150}$$

In view of Lemma 10, Corollary 9 and Theorem 2, the operator $\tilde{K}$ maps $H_e^s(2\pi)$ into $H_e^{s+1}(2\pi)$ and is therefore compact from $H_e^s(2\pi)$ into $H_e^s(2\pi)$. The proof is now complete. □

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A Proof of Lemma 13

Proof. Assuming $\varphi \in C^\infty(\Gamma)$, we define the weighted double layer potential by

$$\mathbf{D}_\omega[\alpha](\mathbf{r}) = \int_\Gamma \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}} \alpha(\mathbf{r}')\omega(\mathbf{r}')d\ell', \quad \mathbf{r} \text{ outside } \Gamma, \tag{151}$$

which, following the related closed-surface calculation presented in [15, Theorem 2.23], we rewrite as

$$\mathbf{D}_\omega[\alpha](\mathbf{r}) = -\mathrm{div}_{\mathbf{r}}\mathbf{E}[\alpha](\mathbf{r}) \tag{152}$$

where

$$\mathbf{E}[\alpha](\mathbf{r}) = \int_\Gamma G_k(\mathbf{r},\mathbf{r}')\alpha(\mathbf{r}')\omega(\mathbf{r}')\mathbf{n}(\mathbf{r}')d\ell'. \tag{153}$$

Since $\mathbf{E}[\alpha] = \mathbf{E} = (E_x, E_y)$ satisfies $\Delta\mathbf{E} + k^2\mathbf{E} = 0$, the two dimensional gradient of its divergence can be expressed in the form

$$\text{grad div}\mathbf{E} = -k^2\mathbf{E} + \left(\frac{\partial}{\partial y}\text{curl }\mathbf{E}, -\frac{\partial}{\partial x}\text{curl }\mathbf{E}\right), \tag{154}$$

where the *scalar rotational* of a two-dimensional vector field $\mathbf{A} = (A_x, A_y)$ is defined by curl $\mathbf{A} = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$. Since $\mathrm{curl}_{\mathbf{r}}\,(\mathbf{n}(\mathbf{r}')G(\mathbf{r},\mathbf{r}'))$ equals $-\mathbf{t}(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')$ (see definition 4), we obtain

$$\mathrm{curl}_{\mathbf{r}}\mathbf{E}[\alpha](\mathbf{r}) = -\int_{-1}^{1} \frac{dG_k(\mathbf{r},\mathbf{r}(t'))}{dt'}\alpha(\mathbf{r}(t'))\omega(\mathbf{r}(t'))dt'. \tag{155}$$

Therefore, taking the gradient of (152), letting $\varphi(t') = \alpha(\mathbf{r}(t'))$, using (21), integrating (155) by parts, and noting that the boundary terms vanish identically (since $\sqrt{1-t'^2} = 0$ for $t' = \pm 1$), we see that

$$\text{grad }\mathbf{D}_\omega[\alpha](\mathbf{r}) = k^2\int_\Gamma G(\mathbf{r},\mathbf{r}')\alpha(\mathbf{r}')\mathbf{n}(\mathbf{r}')\omega(\mathbf{r}')d\ell' - \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}\right), \tag{156}$$

where

$$A(\mathbf{r}) = \int_{-1}^{1} G(\mathbf{r},\mathbf{r}(t'))\frac{d}{dt'}\big(\varphi(t')\sqrt{1-t'^2}\big)dt'. \tag{157}$$

In view of the continuity of the tangential derivatives of single layer potentials across the integration surface (e.g. [15, Theorem 2.17]), in the limit as $\mathbf{r} \to \mathbf{r}(t) \in \Gamma$ we obtain

$$\left(\frac{\partial A(\mathbf{r}(t))}{\partial y}, -\frac{\partial A(\mathbf{r}(t))}{\partial x}\right)\cdot\mathbf{n}(\mathbf{r}(t)) = -\frac{1}{\tau(t)}\frac{dA(\mathbf{r}(t))}{dt}, \tag{158}$$

and the decomposition (133) results. □

B Asymptotic behavior of $\mathbf{NS}[1]$

In this section we demonstrate the poor quality of the composition $\mathbf{NS}$ of the unweighted hypersingular and single-layer operators by means of an example: we consider the flat arc $[-1,1]$ at zero frequency ($\mathbf{NS} = \mathbf{N}_0\mathbf{S}_0$). In detail, in Section B.1 we show that the image of $\mathbf{S}$ is not contained in the domain of $\mathbf{N}$ (and, thus, the formulation $\mathbf{NS}$ cannot be placed in the functional framework [34, 35, 37]), and in Section B.2 we study the edge asymptotics of the function $\mathbf{NS}[1]$ which show, in particular, that the function 1, (which itself lies in $H^s[-1,1]$ for arbitrarily large values of s) is mapped by the operator $\mathbf{NS}$ into a function which does not belong to the Sobolev space $H^{-\frac{1}{2}}[-1,1]$, and, thus, to any space $H^s[-1,1]$ with $s \geq -1/2$.

We thus consider the *unweighted* single-layer and hypersingular operators which, in the present flat-arc, zero-frequency case take particularly simple forms. In view of (6), the *parameter-space form* of the unweighted single-layer operator (which is defined in a manner analogous to that inherent in equation (22) and related text) is given by

$$S_0[\varphi](x) = -\frac{1}{2\pi}\int_{-1}^{1} \ln|x-s|\varphi(s)ds. \tag{159}$$

With regards to the parameter-space form N_0 of the hypersingular operator (7) we note, with reference to that equation, that in the present zero-frequency flat-arc case we have $\mathbf{r} = (x,0)$, $z\mathbf{n_r} = (0,z)$ and $-d/d\mathbf{n_{r'}} = d/dz$. Since, additionally, the single layer potential (2) yields a solution of the Laplace equation in the variables (x,z), we have

$$4\pi N_0[\varphi](x) = -\lim_{z\to 0}\frac{d^2}{dx^2}\int_{-1}^{1}\varphi(s)\ln((x-s)^2+z^2)ds, \tag{160}$$

or equivalently,

$$N_0[\varphi](x) = \frac{1}{4\pi}\lim_{z\to 0}\frac{d}{dx}\int_{-1}^{1}\varphi(s)\frac{d}{ds}\ln((x-s)^2+z^2)ds. \tag{161}$$

Note that, in view of the classical regularity theory for the Laplace equation, letting z tend to zero for $-1 < x < 1$ in equation of (160) we also obtain, for smooth φ,

$$N_0[\varphi](x) = \frac{d^2}{dx^2}S_0[\varphi](x), \quad -1 < x < 1. \tag{162}$$

B.1 The operator S_0

Integrating (159) by parts we obtain

$$-2\pi S_0[1](x) = \int_{-1}^{1}\frac{d(s-x)}{ds}\ln|s-x|ds = (1-x)\ln(1-x) + (1+x)\ln(1+x) - 2, \tag{163}$$

and therefore

$$S_0[1](x) = \frac{1}{2\pi}\big(2-(1-x)\ln(1-x)-(1+x)\ln(1+x)\big). \tag{164}$$

Incidentally, this expression shows that the unweighted single-layer operator does not map C^∞ functions into C^∞ functions up to the edge; a more general version of this result is given in [36, p. 182].

The following two lemmas provide details on certain mapping properties of the operator S_0 .

Lemma 17 *The image $S_0[1]$ of the constant function* 1 *by the operator* (159) *is an element of* $H^{\frac{1}{2}}[-1,1]$.

Proof. Let Γ_1 be a closed, smooth curve which includes the segment $[-1,1]$. Clearly, the function

$$f_1(s) = \begin{cases} 1, & s \in [-1,1] \\ 0, & s \in \Gamma_1 \backslash [-1,1] \end{cases} \tag{165}$$

belongs to $L^2(\Gamma_1)$ and therefore to $H^{-\frac{1}{2}}(\Gamma_1)$, so that, according to Definition 1, the constant 1 is in the space $\tilde{H}^{-\frac{1}{2}}[-1,1]$. In view of equation (10), it follows that $S_0[1] \in H^{\frac{1}{2}}[-1,1]$. □

Lemma 18 *The image $S_0[1]$ of the constant function* 1 *by the operator* (159) *is not an element of* $\tilde{H}^{\frac{1}{2}}[-1,1]$.

Proof. In view of (164) and the fact that $S_0[1](x)$ is an even function of x, integration by parts yields

$$\int_{-1}^{1} e^{-i\xi x} S_0[1](x)dx = \frac{2\sin\xi}{\xi} S_0[1](1) + \frac{1}{2\pi i\xi} \int_{-1}^{1} e^{-i\xi x} \ln\frac{1-x}{1+x} dx. \tag{166}$$

Taking into account the identities [19, eq. 4.381, p 577]

$$\begin{cases} \int_0^1 \ln x \cos\xi x \, dx & = -\frac{1}{\xi}\left[\mathrm{si}(\xi) + \frac{\pi}{2}\right], \\ \int_0^1 \ln x \sin\xi x \, dx & = -\frac{1}{\xi}\left[\mathbf{C} + \ln\xi - \mathrm{ci}(\xi)\right], \end{cases} \qquad \xi > 0 \tag{167}$$

where $\mathbf{C}$ is the Euler constant, and where $\mathrm{si}(\xi)$ and $\mathrm{ci}(\xi)$ are the sine and cosine integrals respectively, (both of which are bounded functions of ξ as $|\xi|$ tends to infinity), it is easily verified that the second term in (166) behaves asymptotically as $\frac{\ln(\xi)}{\xi^2}$ as ξ tends to infinity. Clearly, the first term of (166) decays as $O(\frac{1}{\xi})$, and therefore

$$\left| \int_{-1}^{1} e^{-i\xi x} \left(S_0[1](x)\right) dx \right|^2 = O\left(\frac{1}{\xi^2}\right), \quad \xi \to \infty. \tag{168}$$

Equation (168) tells us that the function $\varphi : \mathbb{R} \to \mathbb{R}$ which equals $S_0[1](x)$ for x in the interval $[-1,1]$ and equals zero in the complement of this interval, does not belong to $H^{\frac{1}{2}}(\mathbb{R})$, and, thus, $S_0[1] \notin \tilde{H}^{\frac{1}{2}}[-1,1]$, as claimed. □

Remark 4 *Lemmas 17 and 18 demonstrate that, as pointed out in Section 1, the formulation* **NS** *of the open-curve boundary-value problems under consideration cannot be placed in the functional framework put forth in [34, 35, 37] and embodied by equations* (10), (11) *and definition 1: the image of the operator* **S** *is not contained in the domain of definition of the operator* **N***; see equations* (10) *and* (11).

B.2 The combination $N_0 S_0$

While, as pointed out in the previous section, $S_0[1]$ does not belong to the domain of definition of N_0 (as set up by the formulation (10), (11)), the quantity $N_0 S_0[1](x)$ can be evaluated pointwise for $|x| < 1$, and it is instructive to study its asymptotics as $x \to \pm 1$.

Lemma 19 *$N_0 S_0[1]$ can be expressed in the form*

$$N_0 S_0[1](x) = \frac{\ln 2 - 1}{\pi^2(1-x^2)} + \mathcal{L}(x), \tag{169}$$

where $\mathcal{L} \in L^2[-1,1]$.

Proof. In view of (164) we have

$$N_0 S_0[1](x) = \frac{1}{\pi} N_0[1](x) - \frac{1}{2\pi} N_0[g](x), \tag{170}$$

where

$$g(x) = (1-x)\ln(1-x) + (1+x)\ln(1+x). \tag{171}$$

For the first term on the right-hand side of this equation we obtain from (162) and (164)

$$N_0[1](x) = -\frac{1}{\pi(1-x^2)}\,. \tag{172}$$

To evaluate the second term $N_0[g]$ in equation (170), in turn, we first integrate by parts equation (161) and take limit as $z \to 0$ and thus obtain

$$\begin{aligned} N_0[g](x) &= \frac{1}{2\pi}\left(\frac{d}{dx}\left(\big[\ln|x-s|g(s)\big]_{-1}^{1}\right) - \frac{d}{dx}\int_{-1}^{1} \ln|x-s|\frac{d}{ds}g(s)ds\right) \\ &= \frac{\ln 2}{\pi}\frac{d}{dx}\left(\ln\left(\frac{1-x}{1+x}\right)\right) + \frac{1}{2\pi}\frac{d}{dx}\int_{-1}^{1} \ln|x-s|\ln\left(\frac{1-s}{1+s}\right)ds, \end{aligned} \tag{173}$$

or

$$N_0[g](x) = \frac{-2\ln 2}{\pi(1-x^2)} - \frac{1}{2\pi} p.v. \int_{-1}^{1} \ln\left(\frac{1-s}{1+s}\right)\frac{1}{s-x}ds. \tag{174}$$

Clearly, to complete the proof it suffices to establish that the functions

$$\mathcal{L}^+(x) = p.v. \int_{-1}^{1} \frac{\ln(1-s)}{s-x}ds \quad \text{and} \quad \mathcal{L}^-(x) = p.v. \int_{-1}^{1} \frac{\ln(1+s)}{s-x}ds \tag{175}$$

are elements of $L^2[-1,1]$.

Let us consider the function $\mathcal{L}^+$ for $x \geq 0$ first. Re-expressing $\mathcal{L}^+(x)$ as the sum of the integrals over the interval $[x-(1-x), x+(1-x)] = [2x-1, 1]$ (which is symmetric respect to x plus the integral over $[-1, 2x-1]$ and using a simple change of variables we obtain

$$\mathcal{L}^+(x) = \int_0^{1-x} \frac{\ln(1-x-u) - \ln(1-x+u)}{u}du + \int_{1-x}^{1+x} \frac{\ln(1-x+u)}{u}du. \tag{176}$$

Letting $z = 1-x$ and $v = \frac{u}{z}$, we see that the first integral in (176) is a constant function of x:

$$\int_0^{z} \frac{\ln(z-u) - \ln(z+u)}{u}du = \int_0^{1} \frac{\ln(1-v) - \ln(1+v)}{v}dv = \text{const.} \tag{177}$$

For the second integral in (176), on the other hand, we write

$$
\begin{aligned}
\int_{1-x}^{1+x} \frac{\ln(1+u-x)}{u} du &= \int_{1-x}^{1+x} \frac{\ln(1+\frac{u}{1-x})}{u} du + \ln(1-x) \int_{1-x}^{1+x} \frac{du}{u} \\
&= \int_{1}^{\frac{1+x}{1-x}} \frac{\ln(1+v)}{v} dv + \ln(1-x) \ln\left(\frac{1+x}{1-x}\right) \\
&= \int_{1}^{\frac{1+x}{1-x}} \frac{\ln(1+v)}{1+v} dv + \int_{1}^{\frac{1+x}{1-x}} \frac{\ln(1+v)}{v(1+v)} dv + \ln(1-x) \ln\left(\frac{1+x}{1-x}\right) \\
&= \frac{1}{2}\left(\ln^2\left(\frac{2}{1-x}\right) - \ln^2 2\right) + \int_{1}^{\frac{1+x}{1-x}} \frac{\ln(1+v)}{v(1+v)} dv + \ln(1-x) \ln\left(\frac{1+x}{1-x}\right).
\end{aligned} \tag{178}
$$

Since the second term on the last line of equation (178) is bounded for $0 \leq x < 1$, it follows that, in this interval, the function $\mathcal{L}^+(x)$ equals a bounded function plus a sum of logarithmic terms and is thus an element of $L^2[0,1]$. Using a similar calculation it is easily shown that $\mathcal{L}^+(x)$ is bounded for $-1 \leq x < 0$, and it thus follows that $\mathcal{L}^+ \in L^2[-1,1]$, as desired. Analogously, we have $\mathcal{L}^- \in L^2[-1,1]$, and the lemma follows. □

Corollary 10 *Let* $\Gamma = [-1,1]$. *Then* $\mathbf{NS}[1]$ *does not belong to the codomain* $H^{-\frac{1}{2}}[-1,1]$ *of the operator* $\mathbf{N}$ *in equation* (11).

Proof. In view of Lemma 19 it suffices to show that the function $h(x) = \frac{1}{1-x^2}$ does not belong to $H^{-\frac{1}{2}}[-1,1]$, or, equivalently, that the primitive $k(x) = -\frac{1}{2}\ln\frac{1-x}{1+x}$ of h does not belong to $H^{\frac{1}{2}}[-1,1]$. Clearly, to establish that $k \notin H^{\frac{1}{2}}[-1,1]$ it suffices to show that the function $\ell(x) = p(x)\ln(x)$ is not an element of $H^{\frac{1}{2}}[0,\infty[$, where p is a smooth auxiliary function defined for $x \geq 0$ which equals 1 in the interval $[0,1]$ and which vanishes outside the interval $[0,2]$.

To do this we appeal to the criterion [23, p. 54]

$$\ell \in H^{\frac{1}{2}}(0,\infty) \iff \ell \in L^2(0,\infty) \quad \text{and} \int_0^\infty t^{-2} dt \int_0^\infty |\ell(x+t) - \ell(x)|^2 \, dx < \infty.$$

To complete the proof of the lemma it thus suffices to show that the integral

$$I = \int_0^\infty t^{-2} dt \int_0^1 |\ln(x+t) - \ln(x)|^2 \, dx$$

is infinite. But, using the change of variables $u = \frac{t}{x}$ we obtain

$$I = \int_0^\infty \frac{1}{t} dt \left(\int_t^\infty \frac{|\ln(1+u)|^2}{u^2} du \right) = \infty, \tag{179}$$

and the lemma follows. □

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